

# 應數一線性代數 2021 春, 期末 考SOLUTION

考試時間：2021/06/24, 09:10 - 12:00,

收卷截止時間：12:10

考卷繳交位置：Google Classroom

## 考試須知:

- 本次是開書限時考，禁止交談討論。開書範圍是你事先準備的紙本/電子檔的作業或筆記，我課程網頁上提供的資訊，紙本/電子檔課本。
- 需要開鏡頭麥克風。鏡頭需要看得到你的身邊，你在作答的紙面，還有你在使用的電子資源的畫面（例如電腦螢幕或平板螢幕）。我不需要直接閱讀螢幕內容，我只要看看畫面的形狀色塊，確定你在看什麼就好。
- 請將紙面答案卷掃成一份 pdf 檔，畫面請清晰並且轉正。第一頁左上寫明姓名學號，每一題前面註明題號，頁面請按照題號順序編排不要跳號。
- 注意事先準備充足的紙張。考試途中不能向外求助更多的計算紙。

1. (5 points) Find the coordinate vector of  $x^3 + 3x^2 - 4x + 3$  in  $P_3$  relative to  $(x^3 - x^2, x^2 - x, x - 1, x^3 + 1)$

From 7-1

$$[-0.5, 2.5, -1.5, 1.5] = \left[-\frac{1}{2}, \frac{5}{2}, -\frac{3}{2}, \frac{3}{2}\right] = \frac{1}{2}[-1, 5, -3, 3]$$

2. (5 points) Express  $(\sqrt{3}i - 1)^8$  in the form  $a + bi$  for  $a, b$  are real numbers. Find  $a, b$ .

From 9-1

$$2^8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -128 - 128\sqrt{3}i$$

3. (10 points) Find the six sixth roots of  $-8i$ . (need not simplify)

From 9-1

$$\sqrt{2}(\cos(45^\circ + 60k^\circ) + i\sin(45^\circ + 60k^\circ)) \text{ or } \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{3}\right)\right), \text{ for } i = 0, 1, 2, 3, 4, 5$$

4. (10 points) Find a vector perpendicular to both  $[0, i, 1 + i]$  and  $[1 + i, 1 - i, 1]$  in  $\mathbb{C}^3$ .

From 9-2

$$[-2 - i, -2i, 1 + i]$$

5. (10 points) Find the matrix representations  $R_{B,B}$ ,  $R_{B',B'}$  and an invertible  $C$  such that  $R_{B',B'} = C^{-1}R_{B,B}C$  for linear transformation  $T: P_2 \rightarrow P_2$  defined by  $T(p(x)) = p(x+1) + p(x)$ ,  $B = (x^2, x, 1)$ ,  $B' = (x^2 + 1, x + 1, 2)$ .

From 7-2

$$R_{B,B} \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}, R_{B',B'} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \\ -0.5 & 0.5 & 2 \end{bmatrix}, C = C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, C_{B,B'} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & -0.5 & 0.5 \end{bmatrix}$$

6. (10 points) Find an unitary matrix  $U$  and a diagonal matrix  $D$  such that  $D = U^{-1}AU$ , where

$$A = \begin{bmatrix} 3 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 3 \end{bmatrix}$$

From 9-3

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = U^{-1}AU = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2}i & 0 & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2}i & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 & -i \\ 0 & 2 & 0 \\ i & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2}i & \frac{\sqrt{2}}{2}i \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

7. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix  $A$ .

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2i & 0 & 0 \\ 0 & 0 & 0 & 2i & 0 \\ 5 & 0 & -1 & 0 & 2i \end{bmatrix}$$

From 9-4

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2i & 1 & 0 \\ 0 & 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & 0 & 2i \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1-2i \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 5 \\ 1-2i \\ 0 \end{bmatrix}$$

$$A\vec{v}_1 = \vec{v}_1, A\vec{v}_2 = \vec{v}_2, A\vec{v}_3 = 2i(\vec{v}_3), A\vec{v}_4 = 2i(\vec{v}_4) + \vec{v}_3, A\vec{v}_5 = 2i(\vec{v}_5)$$

8. (10 points) Answer the following question.

1. Find the eigenvalues of the given Matrix  $J$ .
2. Give the rank and nullity of  $(J - \lambda)^k$  for each eigenvalue  $\lambda$  of  $J$  and for every positive integer  $k$ .
3. Draw schemata of the strings of vectors in the standard basis arising from the Jordan blocks in  $J$ .
4. For each standard basis vector  $\vec{e}_k$ , express  $J\vec{e}_k$  as a linear combination of vectors in the standard basis.

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \end{bmatrix} = J = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & & & & & & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & & & & & & \\ & & \boxed{\begin{matrix} i & 1 & 0 \\ 0 & i & 1 \\ 0 & 0 & i \end{matrix}} & & & & & & \\ & & & \boxed{2} & & & & & \\ & & & & \boxed{i} & & & & \\ & & & & & & & & \end{bmatrix}$$

From 9-4

1.  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_8 = 2, \lambda_5 = \lambda_6 = \lambda_7 = \lambda_9 = i$
2.  $(J - 2I)$  has rank 6 and nullity 3,  
 $(J - 2I)^k$  has rank 4 and nullity 5, for  $k \geq 2$ ,  
 $(J - iI)$  has rank 7 and nullity 2,  
 $(J - iI)^2$  has rank 6 and nullity 3,  
 $(J - iI)^k$  has rank 5 and nullity 4 for  $k \geq 3$ ,
3. The strings are:  $(J - 2I) : \begin{cases} \vec{e}_2 \rightarrow \vec{e}_1 \rightarrow 0 \\ \vec{e}_4 \rightarrow \vec{e}_3 \rightarrow 0 \\ \vec{e}_8 \rightarrow 0 \end{cases}, (J - iI) : \begin{cases} \vec{e}_7 \rightarrow \vec{e}_6 \rightarrow \vec{e}_5 \rightarrow 0 \\ \vec{e}_9 \rightarrow 0 \end{cases}$
4.  $\begin{cases} J\vec{e}_1 = 2\vec{e}_1, \\ J\vec{e}_2 = 2\vec{e}_2 + \vec{e}_1, \end{cases}, \begin{cases} J\vec{e}_3 = 2\vec{e}_3, \\ J\vec{e}_4 = 2\vec{e}_4 + \vec{e}_3, \end{cases}, \begin{cases} J\vec{e}_5 = i\vec{e}_5 \\ J\vec{e}_6 = i\vec{e}_6 + \vec{e}_5, \\ J\vec{e}_7 = i\vec{e}_7 + \vec{e}_6, \end{cases}, \begin{cases} J\vec{e}_8 = 2\vec{e}_8, \\ J\vec{e}_9 = i\vec{e}_9 \end{cases}$

9. (5 points) Prove or disprove the following: For a square matrix  $A$ , we have  $\det(A^*) = \det(A)$

From 9-2 #33 (i)

$$A = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \det(A) = i \neq \det(A) = -i$$

10. (5 points) Prove or disprove the following: If  $U$  is unitary, the  $\overline{U}$  also an unitary matrix.

From 9-2 #42

Let  $U$  is an unitary matrix, i.e.  $U^*U = I = UU^*$ .  $(\overline{U})^*\overline{U} = \overline{U^*}^T \overline{U} = U^T \overline{U} = \overline{U^*U} = \overline{I} = I$ . Similarly,  $\overline{U}(\overline{U})^* = I$ . Hence  $\overline{U}$  is a unitary matrix.

11. (10 points) Find all the possible  $a, b, z \in \mathbb{C}$  such that matrix  $\begin{bmatrix} z & a \\ b & z \end{bmatrix}$  is unitarily diagonalizable.

From 9-3 #14

$$M = \begin{bmatrix} z & a \\ b & z \end{bmatrix}$$

$$MM^* = \begin{bmatrix} z & a \\ b & z \end{bmatrix} \begin{bmatrix} \bar{z} & \bar{b} \\ \bar{a} & \bar{z} \end{bmatrix} = \begin{bmatrix} |z|^2 + |a|^2 & \bar{b}z + a\bar{z} \\ b\bar{z} + \bar{a}z & |b|^2 + |z|^2 \end{bmatrix} = M^*M = \begin{bmatrix} \bar{z} & \bar{b} \\ \bar{a} & \bar{z} \end{bmatrix} \begin{bmatrix} z & a \\ b & z \end{bmatrix} = \begin{bmatrix} |z|^2 + |b|^2 & \bar{b}z + a\bar{z} \\ b\bar{z} + \bar{a}z & |a|^2 + |z|^2 \end{bmatrix}$$

$M$  is unitarily diagonalizable if  $|a| = |b|$

12. (10 points) Show that the  $n^{th}$  roots of  $z \in \mathbb{C}$  can be represented geometrically as  $n$  equally spaced points on the circle  $x^2 + y^2 = |z|^2$ .

From 9-1 #26

The  $n^{th}$  roots of  $z = r(\cos \theta + i \sin \theta)$  are given by

$$r^{\frac{1}{n}} \left( \cos \left( \frac{\theta + 2k\pi}{n} \right) + i \sin \left( \frac{\theta + 2k\pi}{n} \right) \right), \text{ for } k = 0, 1, 2, \dots, n-1.$$

Since the points  $\left\{ \frac{\theta + 2k\pi}{n} \mid k = 0, 1, 2, \dots, n-1 \right\}$  divide the interval  $[0, 2\pi]$  into  $n$  subintervals of equal width  $\frac{2\pi}{n}$  we see that the  $n^{th}$  roots of  $z$  are equally spaced points on the circle  $x^2 + y^2 = (r^{\frac{1}{n}})^2 = (\sqrt[n]{|z|})^2$ .