應數一線性代數 2021 秋, 期末考 解答

學號: ______________________, 姓名: ______________________

本次考試共有 7 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
宏: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________

1. (5 points) Find $\mathbf{v}_B$, which is the coordinate vector of the given vector $\mathbf{v} = [1, 2, -2]$ relative to the indicated ordered basis $\mathbf{B} = \{[1, 1, 1], [1, 2, 0], [1, 0, 1]\}$.

   Ch 3-3 example 2

   $\vec{v}_B = [-4, 3, 2]$

2. (10 points) Find the coordinate vector of polynomial $4x^3 + 5x^2 + x$ relative to the ordered basis $B = ((x + 1)^3, (x + 1)^2, (x + 1), 1)$ of the vector space P_3 of polynomials of degree at most 3.

   Let $\vec{p} = 4x^3 + 5x^2 + x$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 3 & 1 & 0 & 0 & 5 \\ 3 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

   $\vec{p}_B = [4, -7, 3, 0]$

3. (10 points) Let $V = sp(e^{2x}, e^{4x}, e^{8x})$, $V' = sp(e^{3x}, e^{7x}, e^{9x})$ are the subspaces of the vector space of all real-valued functions with domain $\mathbb{R}$, and let $B = (e^{2x}, e^{4x}, e^{8x})$, $B' = (e^{3x}, e^{7x}, e^{9x})$. Let $T : V \longrightarrow V'$ be the linear transformation having the given matrix A as matrix representation relative to B, B'.

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix},$$

Find $T(ae^{2x} + be^{4x} + ce^{8x}) =$ $(2a+2b)e^{3x} + (3b+c)e^{7x} + (-2a+b+3c)e^{9x}$

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+2b \\ 3b+c \\ -2a+b+3c \end{bmatrix}$$

4. (10 points) Suppose that A is a 4×4 matrix with determinant 3.

(a) Find $\det(5A) =$ $3 * 5^4 = 1875$

(b) Find $\det(A^{-1}) =$ $1/3$

(c) Find $\det(2A^{-1}) =$ $2^4/3 = 16/3$

(d) Find $\det((2A)^{-1}) =$ $1/(3 * 2^4) = 1/48$

5. (10 points) Suppose that A is a 3×3 matrix with row vectors $\vec{a}, \vec{b}$, and $\vec{c}$, and that $det(A) = 3$. Find the determinant of the matrix having $2\vec{a} + 3\vec{b} + 2\vec{c}, \vec{c}, \vec{a}$ as its row vectors

Determinant = 9

$$\begin{vmatrix} 2\vec{a}+3\vec{b}+2\vec{c} \\ \vec{c} \\ \vec{a} \end{vmatrix} = \begin{vmatrix} 3\vec{b}+2\vec{c} \\ \vec{c} \\ \vec{a} \end{vmatrix} (R_1 \to R_1 - 2 \times R_3)$$

$$= \begin{vmatrix} 3\vec{b} \\ \vec{c} \\ \vec{a} \end{vmatrix} (R_1 \to R_1 - 2 \times R_2)$$

$$= 3 \begin{vmatrix} \vec{b} \\ \vec{c} \\ \vec{a} \end{vmatrix} (R_1 \to \frac{1}{3} R_1)$$

$$= -3 \begin{vmatrix} \vec{a} \\ \vec{c} \\ \vec{b} \end{vmatrix} (R_1 \leftrightarrow R_3)$$

$$= 3 \begin{vmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{vmatrix} (R_2 \leftrightarrow R_3)$$

$$= 3 \times 3 = 9$$

6. (10 points)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}$$

The inverse of A = $\begin{bmatrix} 0.6 & 0.2 & -0.1 \\ -0.8 & 0.4 & 0.3 \\ 0.4 & -0.2 & 0.1 \end{bmatrix}$, and the adjoint matrix of A = $\begin{bmatrix} 6 & 2 & -1 \\ -8 & 4 & 3 \\ 4 & -2 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 0.6 & 0.2 & -0.1 \\ -0.8 & 0.4 & 0.3 \\ 0.4 & -0.2 & 0.1 \end{bmatrix}, \det(A) = 10$$

7. (5 points) Let $\vec{a} = \vec{i} - 2\vec{k}$, $\vec{b} = -\vec{i} + 3\vec{j}$, $\vec{c} = \vec{i} + 2\vec{j} + \vec{k}$.

   Find $\vec{a} \cdot (\vec{b} \times \vec{c}) =$ **13**

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{a} \cdot [3, 1, -5] = 13$$

8. (10 points) Find out whether points (0, 0, 0, 0) (2, 1, 0, 0), (3, -2, 0, 0), (0, 0, 2, 6) and (0, 0, 2, 3) lie in a plane in $\mathbb{R}^4$.

   Answer: the given 4 points are NOT lie in a plane.

   Assume points $O = (0, 0, 0, 0)$, $A = (2, 1, 0, 0)$, $B = (3, -2, 0, 0)$, $C = (0, 0, 2, 6)$ and $D = (0, 0, 2, 3)$.

$$\overrightarrow{OA} = [2, 1, 0, 0],\ \overrightarrow{OB} = [3, -2, 0, 0],\ \overrightarrow{OC} = [0, 0, 2, 6],\ \overrightarrow{OD} = [0, 0, 2, 3]$$

   The volume generated by $\vec{(}OA)$, $\vec{(}OB)$, $\vec{(}OC)$, $\vec{(}OD)$ is

$$\begin{vmatrix} 2 & 1 & 0 & 0 \\ 3 & -2 & 0 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} \cdot \begin{vmatrix} 2 & 6 \\ 2 & 3 \end{vmatrix} = (-4-3)(6-12) = 42 \neq 0$$

   Hence the given 4 points are NOT lie in a plane.

9. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T([x, y]) = [y, x, 3x + y]$. Let D be the rectangle $2 \leq x \leq 3$, $-1 \leq y \leq 4$. FInd the area of $T(D)$.

   Answer: $5\sqrt{11}$.

   Area of D is $(3 - 2) * (4 - (-1)) = 5$.

   The standard matrix representation of T is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\sqrt{\det(A^T A)} = \sqrt{11}$$

   The area of $T(D) = 5\sqrt{11}$

10. (10 points) Circle True or False. **Write down the explanation if the statement is False.** Read each statement in original Greek before answering.

   (a) **True** False There's an unique coordinate vector associated with each vector $\vec{v} in V$ relatve to an ordered basis for V

   (b) True **False** For every vector $\vec{b}'$ in V', the function$T_{\vec{b}'} : V \to V'$ defined by $T_{\vec{b}'}(\vec{v}) = \vec{b}'$ for all $\vec{v}$ in V is a linear transformation.

   (c) True **False** The parallelogram (平行四邊形) in $\mathbb{R}^2$ determined by non-zero vectors $\vec{a}, \vec{b}$ is a square (正方形) if and only if $\vec{a} \cdot \vec{b} = 0$

   (d) True **False** The product of a square matrix and its adjoint is the identity matrix.

   (e) **True** False There is no square matrix A such that $\det(A^T A) = -1$.

   (b) $T_{\vec{b}'}(\vec{v}) = \vec{b}'$, $T_{\vec{b}'}(2\vec{v}) = \vec{b}' \neq 2T_{\vec{b}'}(\vec{v}) = 2\vec{b}'$

   (c) $\vec{a} = [1, 0], \vec{b} = [0, 2] \Rightarrow \vec{a} \cdot \vec{b} = 0$, but it's not a square.

   (d) By Theorem 4.6: $A * adj(A) = \det(A)I$

11. (10 points) Let $S = \{1, \sin(x), \cos(x), \sin(2x), \cos(2x)\}$ is a set of functions in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

   (a) Prove that S is an independent set in F.

   (b) Find a basis for the subspace of F generated by the functions $\{f_1, f_2, f_3, f_4\}$, where

$$f_1(x) = 1 + 2\sin(x) + \cos(2x) \quad f_2(x) = -2 + 4\sin(x) - \cos(x) + \cos(2x)$$

$$f_3(x) = 7 - 2\sin(x) + 2\cos(x) + \cos(2x) \quad f_4(x) = 4 + \sin(x) + \cos(x) + 2\sin(2x) - \cos(2x)$$

(a) Assume there exists $a, b, c, d, e \in \mathbb{R}$ such that $a + b\sin(x) + c\cos(x) + d\sin(2x) + e\cos(2x) = 0$.

$$\begin{array}{lccccccccccccl}
\boxed{x=0}, & a & + & b\sin(0) & + & c\cos(0) & + & d\sin(2\cdot 0) & + & e\cos(2\cdot 0) & = & 0 & \\
\Rightarrow & a & + & 0 & + & c & + & 0 & + & e & = & 0 & —(1) \\
\boxed{x=\pi}, & a & + & 0 & - & c & + & 0 & + & e & = & 0 & —(2) \\
\boxed{x=\frac{\pi}{2}}, & a & + & b & + & c & + & 0 & - & e & = & 0 & —(3) \\
\boxed{x=\frac{3\pi}{2}}, & a & - & b & + & c & + & 0 & - & e & = & 0 & —(4)
\end{array}$$

by (1), (2), we have $c = 0$, $\Rightarrow a + e = 0$ —(5)

by (3), (4), we have $b = 0$, $\Rightarrow a - e = 0$ —(6)

by (5), (6), we have $a = e = 0$.

Since $a = b = c = e = 0$, we have $d\sin(2x) = 0$ for all x, $\Rightarrow d = 0$.

Hence, we have $a = b = c = d = e = 0$. Therefore, its independent.

(b)

$$\begin{bmatrix} 1 & -2 & 7 & 4 \\ 2 & 4 & -2 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the basis is $\{f_1, f_2, f_4\}$

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	5	10	10	10	10	10	5	10	10	10	10	100
Score:												