應數一線性代數 2022 春, 期中考 解答

學號: ________________________, 姓名: ________________________

本次考試共有 11 頁 (包含封面)，有 10 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) (a) Solve the system $\begin{cases} x_1' = 3x_1 - 5x_2 \\ x_2' = 2x_2 \end{cases}$

   (b) Find the solution that satisfies the initial condition $x_1(0) = 2, x_2(0) = 5$.

   Answer: (a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5k_1e^{2t} + k_2e^{3t} \\ k_1e^{2t} \end{bmatrix}$, (b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 25e^{2t} - 23k_2e^{3t} \\ 5e^{2t} \end{bmatrix}$

   Follow 課本 5-3 example 3

   Follow 109-2 midterm problem 1.

2. (10 points) Let

$$A = \begin{bmatrix} 9 & -3 & 3 \\ -2 & 10 & 2 \\ 1 & 1 & 11 \end{bmatrix}$$

Find (if exists) an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$. Also, find the eigenvalues of A^{100}.

(1) Is A diagonalizable? YES! . If A diagonalizable, C= $\begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$, D= $\begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

(2) The eigenvalue of A are $6, 12$. The eigenvalue of A^{100} are $6^{100}, 12^{100}$.

Follow 課本 5-2 example 3, 4 and Theorem 5.1

Follow 108-2 midterm problem 1.

3. (10 points) Find the formula for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects in the line $3x + 2y = 0$.

   Answer: $T([x, y]) = \underline{\frac{-1}{13}[5x + 2y, 12x - 5y]}$

   Follow 課本 5-2 example 2

   Follow 109-2 midterm problem 2.

4. (10 points) Find all the possible a, b, c, d, x, y so that the matrix A is orthogonal.

$$A = \begin{bmatrix} a & y & 0 \\ 2x & 3y & c \\ x & b & d \end{bmatrix}$$

Let

$$\vec{v}_1 = \begin{bmatrix} a \\ 2x \\ x \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} y \\ 3y \\ b \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix}$$

Since the matrix A is orthogonal, we have $0 = \vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1 \cdot \vec{v}_3 = \vec{v}_2 \cdot \vec{v}_3$ and $1 = |\vec{v}_1|, |\vec{v}_2|, |\vec{v}_3|$ is zero vector.

In the case of $x = 0$. Since $1 = |\vec{v}_1|$, we have $a = 1$. Using $0 = \vec{v}_1 \cdot \vec{v}_2 = ay = y$, we have $y = 0$ and $b = 1$ ($1 = |\vec{v}_2|$). Thus $c = 1, d = 0$. Therefore, we have one solution $[a, b, c, d, x, y] = [1, 1, 1, 0, 0, 0]$

In the case of $x \neq 0$. By $0 = \vec{v}_1 \cdot \vec{v}_3$, we have $2cx + dx = 0$. Thus $c : d = 1 : -2$. Since $1 = |\vec{v}_3|$, $\vec{v}_3 = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix} = \frac{\pm 1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$.

Since $0 = \vec{v}_2 \cdot \vec{v}_3$ and $c : d = 1 : -2$, $3y : b = 2 : 1$. Thus $b = \frac{3y}{2}$ and $\vec{v}_2 = \begin{bmatrix} y \\ 3y \\ 3y/2 \end{bmatrix} = \frac{\pm 1}{7} \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$.

Since $0 = \vec{v}_1 \cdot \vec{v}_2 = 2a + 12x + 3x$, we have $a = \frac{15x}{2}$. Thus $\vec{v}_1 = \begin{bmatrix} \frac{15x}{2} \\ 2x \\ x \end{bmatrix} = \frac{\pm 1}{\sqrt{245}} \begin{bmatrix} 15 \\ 4 \\ 2 \end{bmatrix}$.

$\boxed{ANSWER:}$ Therefore, we have the solution:

$$\vec{v}_1 = \begin{bmatrix} a \\ 2x \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} y \\ 3y \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

or

$$\vec{v}_1 = \begin{bmatrix} a \\ 2x \\ x \end{bmatrix} = \frac{\pm 1}{\sqrt{245}} \begin{bmatrix} 15 \\ 4 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} y \\ 3y \\ b \end{bmatrix} = \frac{\pm 1}{7} \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ c \\ d \end{bmatrix} = \frac{\pm 1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

5. (10 points) Find the projection matrix P for the plane $W : 2x - y + 2z = 0$ and then find the projection of $\vec{b} = [3, 2, 1]$ on the plane.

Answer: $\vec{b}_W = \underline{\frac{1}{3}[5, 8, -1]}$, $P = \underline{\frac{1}{9}\begin{bmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{bmatrix}}$.

Follow 課本 6-4 example 3

Follow 109-2 midterm problem 4.

6. (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Factor A in the form $A = QR$, where Q is an orthogonal matrix and R is an upper-triangular invertible matrix.

Answer: Q=$\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & \sqrt{2} & 1 \\ 0 & \sqrt{2} & -2 \\ -\sqrt{3} & \sqrt{2} & 1 \end{bmatrix}$, R=$\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{3} & \sqrt{3}/3 \\ 0 & 0 & \frac{2}{3}\sqrt{6} \end{bmatrix}$.

Follow 課本 6-2 example 5.

Follow 109-2 midterm problem 6.

7. (10 points) Find the lease squares straight line fit to the five points (-4, -2), (-2, 0), (0,1), (2, 4), (4, 5) and use it to approximate the fifth points (1, a).

   Answer: the line equation = $0.9x + 1.6$, a= 2.5 .

   Follow 課本 6-5 example 1

8. (10 points) Prove that, for every square matrix A all of whose eigenvalues are real, the product of its eigenvalues is $\det(A)$

   Section 5-2, problem 17

   If the characteristic polynomial of A is $p(\lambda) = |A - \lambda I|$, then $p(0) = |A| = \det(A)$.
   Also,

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

   , so

$$p(0) = (-1)^{2n}\lambda_1\lambda_2\cdots\lambda_n = \lambda_1\lambda_2\cdots\lambda_n = \det(A).$$

9. (10 points) Show that the real eigenvalue of an orthogonal matrix must be equal to 1 or -1.

   Hint: Think in terms of linear transformations.

   Section 6-3, problem 27

10. (10 points) Circle True or False and disprove the statement if it is FALSE. Read each statement in original Greek before answering.

(a) True **False** A square matrix is orthogonal if its column vectors are orthogonal.

Section 6-3, problem 19a. $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ has orthogonal column vectors but A is not orthogonal.

(b) True **False** Every invertible matrix is diagonalizable.

Section 5-2, problem 13f. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. A is invertible but not diagonalizable.

(c) True **False** The intersection of W and $W^{\perp}$ is empty.

Section 6-1, problem 23h. The intersection of W and $W^{\perp}$ is $\{\vec{0}\}$

(d) True **False** The least-square solution vector of $A\vec{x} = \vec{b}$ is the projection of $\vec{b}$ on the column space of A.

Section 6-5, problem 21f. The least-square solution vector of $A\vec{x} = \vec{b}$ is the vector $\vec{y}$ such that $A\vec{y}$ is the projection of $\vec{b}$ on the column space of A.

(e) True **False** If λ is an eigenvalue of a matrix A, then λ is an eigenvalue of $A + cI$ for all nonzero scalar c.

Section 5-1, problem 23g. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. 1 is an eigenvalue of a matrix A, but not an eigenvalue of a matrix $A + 2I = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											