應數一線性代數 2022 秋, 第一次期中考 解答

學號: ______________________, 姓名: ______________________

本次考試共有 8 頁 (包含封面)，有 12 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
宏: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________

1. (10 points) (a) Find the inverse of the matrix A, if it exists, and (b) express the inverse matrix as a product of elementary matrices. $A = \begin{bmatrix} 5 & 2 \\ 3 & 8 \end{bmatrix}$

Answer: (a) $A^{-1} = \frac{1}{34}\begin{bmatrix} 8 & -2 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{17} & \frac{-1}{17} \\ \frac{-3}{34} & \frac{5}{34} \end{bmatrix}$, (b) $A^{-1} = \begin{bmatrix} 1 & -2/5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5/34 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3/5 & 1 \end{bmatrix}$

2. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T([1,0,0]) = [2,4,0], T([0,1,0]) = [1,0,3]$, and $T([2,1,3]) = [11,23,3]$. Find $T([4,-3,2]) =$ $[9,\ 26,\ -9]$

Solution：

$T([0,0,1]) = \frac{1}{3}(T([1,2,3]) - 2T([1,0,0]) - T([0,1,0])) = [2,5,0]$ Let A is the standard matrix representation of T.

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix}$$

$$T([4,-3,2]) = (A\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix})^T = [9,\ 26,\ -9]$$

3. (5 points) Find all possible scalar c such that the vector $\vec{i} + c\vec{j} - 3\vec{k}$ is in the span of $\vec{i} + \vec{j} - \vec{k}$ and $\vec{j} + 3\vec{k}$.

   Answer: $c =$ **1/3** .

   Solution：

   There's integers a, b such that $\vec{i} + c\vec{j} - 3\vec{k} = a(\vec{i} + \vec{j} - \vec{k}) + b(\vec{j} + 3\vec{k})$.

   Therefore, $a = 1$ and $(\vec{i} + c\vec{j} - 3\vec{k}) - (\vec{i} + \vec{j} - \vec{k}) = b(\vec{j} + 3\vec{k})$. Thus $b = -2/3$ and $c = 1/3$

4. (5 points) Given two vectors $\vec{v} = [4, x, 2, 1]$ and $\vec{u} = [8, 2, 4, y]$. Find all $x, y \in \mathbb{R}$ so that

   (a) $\vec{v}, \vec{u}$ are parallel. $x = 1,\ y = 2$.

   (b) $\vec{v}, \vec{u}$ are perpendicular. $x \in \mathbb{R},\ y = -2x - 40$.

   Solution：

   (a) Since the first component of $\vec{v}$ is 4 and the first component of $\vec{u}$ is 8, we know that $\vec{v}, \vec{u}$ are parallel if $2\vec{v} = \vec{u}$. Thus $x = 1, y = 2$.

   (b) It is fact that $\vec{v}, \vec{u}$ perpendicular if $\vec{v} \cdot \vec{u} = 0$. Since $\vec{v} \cdot \vec{u} = 32 + 2x + 8 + y$, $\vec{v}, \vec{u}$ perpendicular if $40 + 2x + y = 0$.

5. (5 points) Suppose that T is a linear transformation with standard matrix representation A, and that A is a 9×15 matrix such that the nullspace of A has dimension 5.

   (a) What is the dimension of the range of T? **10** .

   (b) What is the dimension of the kernel of T? **5** .

   Solution：

   Since the nullity of A is equal to 5, the rank of A is equal to 15-5=10. Thus the dimension of the kernel of T is 5, and the dimension of the range of T is 10.

6. (10 points) Consider the given linear system:

$$\begin{cases} x_1 - x_2 \qquad + \ x_4 = 1 \\ x_1 \qquad - x_3 + 2x_4 = 0 \\ \quad - x_2 + x_3 + \ x_4 = -6 \end{cases}$$

(a) Write its associated augmented matrix. $\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 1 & -6 \end{array}\right]$

(b) Reduce the matrix to its reduced row-echelon form (rref). $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 7 \\ 0 & 1 & -1 & 0 & 5/2 \\ 0 & 0 & 0 & 1 & -7/2 \end{array}\right]$.

(c) Find the homogeneous solution of the linear system . $\left\{ r\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$.

(d) Find the general solution of the linear system . $\left\{ \begin{bmatrix} 7 \\ 5/2 \\ 0 \\ -7/2 \end{bmatrix} + r\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$.

Solution：

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 2 & 0 \\ 0 & -1 & 1 & 1 & -6 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 7 \\ 0 & 1 & -1 & 0 & 5/2 \\ 0 & 0 & 0 & 1 & -7/2 \end{array}\right]$$

Let $x_3 = r$, we get $x_4 = -7/2, x_2 - r = 5/2, x_1 - r = 7$. Thus $x_1 = 7 + r, x_2 = 5/2 + r, x_3 = r, x_4 = -7/2$. Then solution are

$$\left\{ \begin{bmatrix} 7+r \\ 5/2+r \\ r \\ -7/2 \end{bmatrix} \middle| r \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 7 \\ 5/2 \\ 0 \\ -7/2 \end{bmatrix} + r\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$$

7. (10 points) Determine if the line $y = mx$ is a subspace of $\mathbb{R}^2$. *Hint:* Write the line as a set $W = \{[x, mx] | x \in \mathbb{R}\}$.

Solution：

1-6 #11

$W = \{[x, mx] \ | \ x \in \mathbb{R}\}$ is nonempty since $[0, 0] \in W$.

1. Let $[x_1, mx_1]$ and $[x_2, mx_2]$ be in W.

$$[x_1, mx_1] + [x_2, mx_2] = [x_1 + x_2, mx_1 + mx_2] = [(x_1 + x_2), m(x_1 + x_2)] \in W$$

2. Let $[x_1, mx_1] \in W$ and $r \in \mathbb{R}$.

$$r[x_1, mx_1] = [rx_1, rmx_1] = [(rx_1), m(rx_1)] \in W$$

Thus W is nonempty and closed under addition and scalar multiplication, so it is a subspace of $\mathbb{R}^2$.

8. (10 points) Assume the the matrix A can be row reduces to H, please answer the following questions.

$$A = \begin{bmatrix} 5 & 3 & 1 & 2 & 19 & 5 \\ 1 & 1 & 1 & 0 & 3 & -1 \\ 0 & 2 & 4 & -1 & -4 & -9 \\ 1 & -1 & -3 & -4 & 7 & 3 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & -1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 0 & -2 & -4 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) the **rank** of matrix A, is ___3___.

(b) a basis for the **row space** of A is ___[5, 0, 0, 5, 0], [0, 2, 0, -4, -6],[0, 0, -1, 0, -2]___.

(c) a basis for the **column space** of A is $\begin{bmatrix} 5 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

(d) a basis for the **nullspace** of A is $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$.

Solution：

Let $x_3 = t, x_5 = r, x_6 = s$, we get $x_1 - t + 5r + 3s = 0,\ x_2 + 2t - 2r - 4s = 0,\ x_4 + s = 0$. Thus $x_1 = t - 5r - 3s, x_2 = -2t + 2r + 4s, x_3 = t, x_4 = -s, x_5 = r, x_6 = s$. Then homogeneous solution are

$$\left\{ \begin{bmatrix} t-5r-3s \\ -2t+2r+4s \\ t \\ -s \\ r \\ s \end{bmatrix} \middle| t, r, s \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -5 \\ 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 4 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \middle| t, r, s \in \mathbb{R} \right\}$$

9. (5 points) Consider the set $\mathbb{R}^2$, with the addition defined by $[x, y] \oplus [a, b] = [x + a + 1, y + b]$, and with scalar multiplication defined by $r \otimes [x, y] = [rx - 1, ry]$.

   a. Is this set a vector space? __**No!**__
      Hint: Show by verifying the closed under two operations, A1-A4 and S1-S4.

   b. What is the zero vector in this vector space? *Hint:* The zero vector may NOT be the vector $[0, 0]$.
      Answer: the zero vector is _____ , for any vectors [x,y], the -[x,y] is _____

Solution：

Assume the given set is a vector space, then the $\vec{0}$ and the additive inverse for $[x, y]$ are

$$\vec{0} = 0 \otimes [x, y] = [-1, 0], \quad -[x, y] = (-1) \otimes [x, y] = [-x - 1, -y]$$

Check the property A4:

$$[x, y] \oplus (-[x, y]) = [x, y] \oplus [-x - 1, -y] = [x + (-x - 1) + 1, y + (-y)] = [0, 0] \neq \vec{0} = [-1, 0]$$

Check the property S4:

$$1 \otimes [x, y] = [x - 1, y] \neq [x, y]$$

Since the properties A4 and S4 are failed, the given set is NOT a vector space!

10. (10 points) For vectors $\vec{u}, \vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$ and for scalars r and s, prove that, if $\vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{u}$, then $\vec{w}$ is perpendicular to $r\vec{u} + s\vec{v}$.

Solution：

1-2 #44

We know that $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$, so $\vec{w} \cdot \vec{u} = 0$ and $\vec{w} \cdot \vec{v} = 0$. Then

$$\begin{aligned} \vec{w} \cdot (r\vec{u} + s\vec{v}) &= \vec{w} \cdot (r\vec{u}) + \vec{w} \cdot (s\vec{v}) \\ &= r(\vec{w} \cdot \vec{u}) + s(\vec{w} \cdot \vec{v}) \\ &= r(0) + s(0) = 0 \end{aligned}$$

Therefore $\vec{w}$ is perpendicular to $r\vec{u} + s\vec{v}$

11. (10 points) Prove that the given relation holds for all matrices for which the expressions are defined.

$$(AB)C = A(BC)$$

Solution：

1-3 #33

12. (10 points) Circle each of the following True or False and then give a counterexample （反例） for the false statement.

1. True **False** For all positive integers m and n, the nullity of an $m \times n$ matrix might be any number from 0 to n.

Solution：
For any 3×8 matrix, it could never have the nullity equals 0.

2. True **False** There are exactly two unit vectors perpendicular (垂直) to any given nonzero vectors in $\mathbb{R}^n$.

Solution：
For $n = 3$, $\vec{e}_1 = [1, 0, 0]$ is perpendicular to every vector in $S = sp(\vec{e}_2, \vec{e}_3) = sp([0, 1, 0], [0, 0, 1])$.

3. True **False** Every independent subset of $\mathbb{R}^n$ is a subsets of every basis for $\mathbb{R}^n$.

Solution：
Let $\mathcal{B} = \{\vec{e}_1, \vec{e}_2, \vec{v} = [1, 1, 1]\}$ and it is a basis for $\mathbb{R}^2$. $\mathcal{S} = \{\vec{e}_2, \vec{e}_3\}$ is an independent subsets of $\mathbb{R}^2$, but $\mathcal{S}$ is NOT a subsets of $\mathcal{B}$.

4. True **False** Every vector spaces has at least two vectors.

Solution：
$\{\vec{0}\}$ with normal vector addition and normal scalar multiplication is a vector space with only one vector.

5. True **False** Every function mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ is a linear transformation.

Solution：
Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T([x, y]) = [\cos(x) + y, 3x, 6y]$. T is NOT a linear transformation since $T(2[\pi, 0]) \neq 2T([\pi, 0])$.

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	10	10	5	5	5	10	10	10	5	10	10	10	100
Score:													