應數一線性代數 2022 秋, 期末考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 9 頁 (包含封面)，有 10 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
宏: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________________

1. (10 points) Find the area of the parallelogram(平行四邊形) in $\mathbb{R}^3$ determined by the vectors $[2, -3, 5]$ and $[4, -5, 1]$

   Answer: area = $\sqrt{812}$

   Solution：

   Let $\vec{a} = [2, -3, 5]$ and $\vec{b} = [4, -5, 1]$, then the area is $\|\vec{a} \times \vec{b}\|$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 4 & -5 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -3 & 5 \\ -5 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 5 \\ 4 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -3 \\ 4 & -5 \end{vmatrix} = [22, 18, 2]$$

   The length of $[22, 18, 2]$ is $\sqrt{812}$

2. (10 points) Find the coordinate vector of the given vector relative to the indicated ordered basis.
$7x^3 + 3x^2 - 2x + 3$ in P_3 relative to $(x^2 + x, x^3 + 2x - 1, x^3 + x, 2x^2 + 1)$
Answer: the coordinate vector is **[-15, 6, 1, 9]**

Solution：

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 0 & 7 \\ 1 & 0 & 0 & 2 & 3 \\ 1 & 2 & 1 & 0 & -2 \\ 0 & -1 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -15 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

3. (10 points) Let $T : P_2 \to P_3$ be defined by $T(p(x)) = (x-2)p(x+1)$, the ordered basis for P_2 is $B = (x^2, x, 1)$ and the ordered basis for P_3 is $B' = (x^3, x^2, x, 1)$. Fine the standard matrix representation A of T relative to the ordered bases B and B'.

Answer: (a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \\ -2 & -2 & -2 \end{bmatrix}$

(b) $T(-2x^2 - 4x + 3) = \underline{-2x^3 - 4x^2 + 13x + 6}$

(c) The $ker(T) = \underline{\{0\}}$

Solution：

$$\begin{aligned} T(x^2) &= (x-2)(x+1)^2 = x^3 - 3x - 2, \\ T(x) &= (x-2)(x+1) = x^2 - x - 2, \\ T(1) &= x - 2 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \\ -2 & -2 & -2 \end{bmatrix}, \; rref(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

By the $rref(A)$, we find the $ker(T)_B = \{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \}$, i.e. $ker(T) = \{0x^2 + 0x + 0 = 0\}$

Let $p(x) = -2x^2 - 4x + 3$, $p(x)_B = \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix}$.

$$T(p)_{B'} = Ap_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -1 & 1 \\ -2 & -2 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 13 \\ 6 \end{bmatrix}$$

$T(p) = -2x^3 - 4x^2 + 13x + 6$

4. (10 points) Let V and V' be vector spaces with ordered bases $B = ([1,3,-2],[4,1,2],[-1,1,0])$ and $B' = ([1,0,1,0],[2,1,1,-1],[0,1,1,-1],[2,0,3,1])$, respectively, and let $T : V \longrightarrow V'$ be the linear transformation having the given matrix A as matrix representation relative to B, B'. For a vector $\vec{v}$ such that $\vec{v}_B = [1,-3,10]$, find $T(\vec{v})$.

$$A = \begin{bmatrix} 0 & 4 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

Answer: If $\vec{v}_B = [1,-3,10]$, then $\vec{v} =$ **[-21, 10, -8]** .
$T(\vec{v}) =$ $[28, 30, 29, -23]$.

Solution：

$$\vec{v} = \begin{bmatrix} 1 & 4 & -1 \\ 3 & 1 & 1 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 10 \end{bmatrix} = \begin{bmatrix} -21 \\ 10 \\ -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 4 & -1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 10 \end{bmatrix} = \begin{bmatrix} 28 \\ 30 \\ 29 \\ -23 \end{bmatrix}$$

5. (10 points) Suppose that A is a 5×5 matrix with determinant 2.

(a) Find $\det(3A) =$ $\underline{3^5 \times 2 = 486}$.

(b) Find $\det(A^{-1}) =$ $\underline{1/2}$.

(c) Find $\det(7A^{-1}) =$ $\underline{7^5/2 = 16807/2 = 8403.5}$.

(d) Find $\det((7A)^T) =$ $\underline{7^5 \times 2 = 33614}$.

6. (10 points)

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 5 & 1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$$

The inverse of A = $\underline{\frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -1 & -1 & 5 \\ 9 & 3 & -15 \end{bmatrix}}$, and the adjoint matrix of A = $\underline{\begin{bmatrix} 2 & 2 & -4 \\ -1 & -1 & 5 \\ 9 & 3 & -15 \end{bmatrix}}$

Solution：

$\det(A) = 6.$

7. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by $T([x, y]) = [y, x, x + y]$. Find the volume of the image under T of the disk $x^2 + y^2 \le 16$.

   Answer: $16\sqrt{3}\pi$.

   Solution：

   The standard matrix representation of T is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\det(A^T A) = \det(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}) = 3$$

   Let $D = \{(x, y) \mid x^2 + y^2 \le 16\}$, which is a disk with radius 4. Therefore, the volume of D is 16π and the volume of $T(D) = \sqrt{\det(A^T A)} \cdot 16\pi = 16\sqrt{3}\pi$.

8. (10 points) Show that $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ for any vectors $\vec{a}, \vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$.

   Solution：

   See section 4-1 problem 59.

9. (10 points) Circle each of the following True or False. Please give a counterexample (反例) for the false statement and give an explain （解釋） for the true statement.

(a) **True** False A linear transformation $T : V \to V'$ carries a pair $\vec{v}, \ -\vec{v}$ in V into a pair $\vec{v}', \ -\vec{v}'$ in V'

Solution：

Let $\vec{v}' = T(\vec{v})$

$$T(-\vec{v}) = T((-1)\vec{v}) = (-1)T(\vec{v}) = -T(\vec{v}) = -\vec{v}'$$

(b) True **False** The product of a square matrix and its adjoint is the identity matrix.

Solution：

$$A \cdot adj(A) = \det(A) \cdot I$$

(c) **True** False There is no square matrix A such that $\det(A^T A) = -1$.

Solution：

$$\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 \neq -1$$

(d) True **False** The determinant of 3×3 matrix is zero if the points in $\mathbb{R}^3$ given by the rows of the matrix lie in a plane.

Solution：

$A : (1, 0, 0), \ B = (0, 1, 0), \ C = (0, 0, 1)$ are lie in the plane $x + y + z = 1$.

However,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

(e) True **False** Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. The image under T of an n-box in $\mathbb{R}^n$ of. volume > 0 is a n-box in $\mathbb{R}^n$ of volume > 0

Solution：

$T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(\vec{v}) = \vec{0}$ is a linear transformation. However, the image under T of any n-box is just $\vec{0}$, which the volume is 0.

p.s. compare this statement in the section 4-4 problem 35(c), they are different.

10. (10 points) (a) Determine the set S_1 of all functions f such that $f(0) = 1$ is a subspace in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

Answer: Is S_1 a subspace of F? **NO**

Solution：

Let $f(x), g(x) \in S_1$, then $(f \oplus g)(0) = f(0) + g(0) = 1 + 1 = 2 \neq 1$. Therefore, $f \oplus g \notin S$ and S_1 is NOT a subspace of F.

(b) Determine the set of all functions f such that $f(1) = 0$ is a subspace in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

Answer: Is S_2 a subspace of F? **Yes**

Solution：

Let $f(x), g(x) \in S_2$, then $(f \oplus g)(1) = f(1) + g(1) = 0 + 0 = 0$. Thus, $f \oplus g \in S$ and S_1.

Let $f(x) \in S_2$ and $r \in \mathbb{R}$, then $(r \otimes f)(1) = r \cdot f(1) = r \cdot 0 = 0$ Thus, $r \otimes f \in S_2$.

Therefore S_2 is a subspace of F.

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											