應數一線性代數 2023 春, 期中考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 10 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Let

$$A = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 7 & -2 \\ 1 & -2 & 4 \end{bmatrix}$$

Find (if exists) an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$. Also, find the eigenvalues of A^{100}.

(1) Is A diagonalizable? **YES!** . If so, C= $\begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$, D_1= $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

If A is not diagonalizable, why? ____ .

Solution：

Follow 課本 5-2 example 3, 4 and Theorem 5.1

Follow 108-2 midterm problem 1.

2. (10 points) Let

$$A = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 7 & -2 \\ 1 & -2 & 4 \end{bmatrix}$$

Find (if exists) an orthogonal matrix C and a diagonal matrix D such that $D = C^{-1}AC$. Also, find the eigenvalues of A^{100}.

(1) Is A orthogonal diagonalizable? If so, $C=$ $\begin{bmatrix} 1/\sqrt(6) & 1/\sqrt(3) & 1/\sqrt(2) \\ -2/\sqrt(6) & 1/\sqrt(3) & 0/\sqrt(2) \\ 1/\sqrt(6) & 1/\sqrt(3) & -1/\sqrt(2) \end{bmatrix}$, $D=$ $\begin{bmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

If A is not diagonalizable, why? ____ .

(2) The eigenvalue of A are $9, 3$. The eigenvalue of A^k are $9^k, 3^k$.

(3) $A^k =$ ____ . (不需化簡)

Solution：

Follow 課本 5-2 example 3, 4 and Theorem 5.1

3. (5 points) Solve the system $\begin{cases} x_1' = 4x_1 - 2x_2 + x_3 \\ x_2' = -2x_1 + 7x_2 - 2x_3 \\ x_3' = x_1 - 2x_2 + 4x_3 \end{cases}$

Answer: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C \begin{bmatrix} k_1e^{9t} \\ k_2e^{3t} \\ k_3e^{3t} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1e^{9t} \\ k_2e^{3t} \\ k_3e^{3t} \end{bmatrix} = \begin{bmatrix} k_1e^{9t} + k_2e^{3t} + k_3e^{3t} \\ -2k_1e^{9t} + k_2e^{3t} \\ k_1e^{9t} + k_2e^{3t} - k_3e^{3t} \end{bmatrix}$.

Solution：

Follow 課本 5-3 example 3

Follow 109-2 midterm problem 1.

4. (10 points) Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Factor A in the form $A = QR$, where Q is an orthogonal matrix and R is an upper-triangular invertible matrix.

Answer: Q= $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{-\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{3} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{3}}{3} & \frac{-\sqrt{6}}{6} \end{bmatrix}$, R= $\begin{bmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{3} & \frac{\sqrt{3}}{3} \\ 0 & 0 & \frac{\sqrt{6}}{3} \end{bmatrix}$.

Solution：

Follow 課本 6-2 example 5.

Follow 111-2 quiz 6.

Follow 109-2 midterm problem 6.

5. (15 points) (1) Find the projection matrix P that project vectors in $\mathbb{R}^3$ on $W = sp([1,0,-1],[1,1,1])$.

(2) Given $\vec{b} = [3,2,1]$, please find the projection $\vec{b}_W$.

(3) If $\vec{b}_W = \alpha[1,0-1] + \beta[1,1,1]$, find α, β.

Answer: $P = \frac{1}{6}\begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$, $\vec{b}_W =$ **[3, 2, 1]**, $\alpha =$ **1**, $\beta =$ **2**.

6. (10 points) Find the formula for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects in the line $5x + 7y = 0$.

   Answer: $T([x, y]) =$ $\frac{-1}{37}[12x - 35y, -35x - 12y]$.

   Solution：

   Follow 課本 5-2 example 2

   Follow 109-2 midterm problem 2.

7. (10 points) Find the least squares straight line fit to the five points (-3, 2.7), (-1, 3.5), (0,4), (1, 4.5), (3, 5.3) and use it to approximate the fifth points (2, a).

   Answer: the line equation = $0.44x + 4$, a= 4.88 .

   Solution：

   Follow 課本 6-5 example 1

8. (10 points) Circle True or False and then prove (證明) or disprove (反駁) it. Read each statement in original Greek before answering. *** **只圈對錯，沒有論述一律不給分** ***

(a) True **False** An $n \times n$ symmetric matrix A is a projection matrix if and only if $A^2 = I$.

Solution：

Section 6-4, problem 15h.

(b) True **False** Every invertible matrix is diagonalizable.

Solution：

Section 5-2, problem 13f.

(c) True **False** There is a unique polynomial fnction of degree k with graph passing through any k points in $\mathbb{R}^2$ having distinct first coordinates.

Solution：

Section 6-5, problem 21b.

(d) True **False** Every vector in a vector space V is an eigenvector of the identity transformation of V into V.

Solution：

Section 5-1, problem 23i

(e) **True** False Given W is a subspace of $\mathbb{R}^n$. If a vector $\vec{v}$ belongs to both W and $W^\perp$, then $\vec{v} = \vec{0}$.

Solution：

上課證過

9. (10 points) Show that the real eigenvalue of an orthogonal matrix must be equal to 1 or -1.

   Hint: Think in terms of linear transformations.

   Solution：

   Section 6-3, problem 27

10. (10 points) Let P be the projection matrix for k-dimensional subspace of $\mathbb{R}^n$. Please find all eigenvalues of P and also find the algebraic multiplicity of each eigenvalues.

    Solution：

    Section 6-4, problem 19

11. (10 points) If λ is an eigenvalue of an invertible matrix A with $\vec{v}$ as a corresponding eigenvector, please prove that $\lambda \neq 0$ and $1/\lambda$ is an eigenvalue of A^{-1}, again with $\vec{v}$ as a corresponding eigenvector.

Solution：

Section 5-1 # 28, 我上課有證過。

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	5	10	15	10	10	10	10	10	10	110
Score:												