應數一線性代數 2023 春, 期末考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 10 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
沒有計算過程，就算回答正確答案也不會得到滿分。
答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Express $\frac{z}{w}$ in the form $a+bi$, where $a, b \in \mathbb{R}$, if

$$z=-1+i, \quad w=3+4i$$

Answer: $\frac{z}{w}=$ $\underline{\frac{1+7i}{25}}$.

Solution：

From 9-1.

2. (10 points) Find the five fifth roots of $\sin(30^\circ)+i\cos(30^\circ)$. (need not simplify)

Solution：

From 9-1.

$\sin(30^\circ)+i\cos(30^\circ)=\cos(60^\circ)+i\sin(60^\circ)=\cos(\frac{\pi}{3})+i\sin(\frac{\pi}{3})$.

$$\left(\cos\left(\frac{\frac{\pi}{3}+2k\pi}{5}\right)+i\sin\left(\frac{\frac{\pi}{3}+2k\pi}{5}\right)\right), \text{ for } i=0,1,2,3,4$$

3. (10 points) Let A is an 3×3 complex matrix with $\det(A)=2+3i$. Please the value for $\det(iA)$ and $\det(A^*)$.

Answer: $\det(iA)=$ $\underline{i^3(2+3i)=3-2i}$, $\det(A^*)=$ $\underline{2-3i}$, $\det(A^2)=$ $\underline{(2+3i)^2=-5+12i}$.

Solution：

From 9-2, using the technique from Sec. 4-2.

A is an $n\times n$, then $\det(aA)=a^n\det(A)$ and $\det(A)=\det(A^T)$.

Moreover, the definition of the conjugate transpose is also requested. If $A=[a_{ij}]$, $A^*=\overline{A}^T=\overline{[a_{ji}]}=[\overline{a_{ji}}]$

4. (10 points) Given the coordinate vector $\vec{v}_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$. Please find the $\vec{v}$ and $\vec{v}'_B$ when the ordered basis B and B' for P_2 are

$$B = (x^2 - x,\ 2x + 1,\ -x - 5),\ B' = (1,\ (2 + x),\ (2 + x)^2)$$

Answer: $\vec{v} =$ $\underline{2x^2 - 7x - 16}$, $\vec{v}'_B =$ $\begin{bmatrix} 6 \\ -15 \\ 2 \end{bmatrix}$

Solution：

From 7-1

$$\vec{v} = M_B \vec{v}_B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -16 \end{bmatrix} \Rightarrow 2x^2 - 7x - 16$$

$$\vec{v}_{B'} = C_{BB'} \vec{v}_B = M_{B'}^{-1} M_B \vec{v}_B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 4 \\ 1 & 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -15 \\ 2 \end{bmatrix}$$

5. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible C such that $R_{B',B'} = C^{-1}R_{B,B}C$ for linear transformation $T : P_2 \to P_2$ defined by $T(p(x)) = \frac{d}{dx}p(x-1)$, $B = (x^2, x, 1)$, $B' = (x^2-1, x-3, 2)$.

$C_{B,B'}=$ $\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, $C_{B',B}=$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$, $R_{B',B'}=$ $\frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 1 & 0 \end{bmatrix}$ and

$R_{B,B}=$ $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}$.

Is C=$C_{B,B'}$ or $C_{B',B}$? $C_{B'B}$.

Solution：

From 7-2

$$T(x^2) = \frac{d}{dx}(x-1)^2 = 2x-2,\ T(x) = \frac{d}{dx}(x-1) = 1,\ T(1) = \frac{d}{dx}1 = 0$$

Thus

$$T(\begin{bmatrix} a \\ b \\ c \end{bmatrix}) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = R_E\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We have

$$R_{B,B} = R_B = R_E = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

By $C_{B',B} = M_B^{-1}M_{B'} = M_E^{-1}M_{B'} = I^{-1}M_{B'} = M_{B'}$,

$$C = C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$C_{B,B'} = C_{B',B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

Since

$$R_{B'} = R_{B',B'} = C_{B,B'}R_BC_{B',B}$$

$$R_{B'} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 4 & 1 & 0 \end{bmatrix}$$

6. (10 points) Find an unitary matrix U and a diagonal matrix D such that $D = U^{-1}AU$. Also find where

$$A = \begin{bmatrix} 2 & 0 & 1-i \\ 0 & -3 & 0 \\ 1+i & 0 & 1 \end{bmatrix}$$

Answer: D = $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, U = $\begin{bmatrix} 0 & \frac{-1+i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Solution：

From 9-3.

D = $\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, C = $\begin{bmatrix} 0 & -1+i & 1-i \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ such that $D = C^{-1}AC$.

Since A is Hermitian matrix, it is unitarily diagonalizable.

We also notice that $\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1+i \\ 0 \\ 2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1+i \\ 0 \\ 2 \end{bmatrix} \right\rangle = 0$

$$U = \begin{bmatrix} 0 & \frac{-1+i}{\sqrt{6}} & \frac{1-i}{\sqrt{3}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

7. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Answer: Jordan canonical form = $J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$,

Jordan basis = $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Solution：

From 9-4. Quiz 16.

8. (10 points) Find a polynomial in A that gives the zero matrix.

$$A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Answer: $(A-4I)^2(A-9iI)^3 = O$ **or** $(A-4I)^4(A-9iI)^5 = O$.

9. (10 points) Prove that every 2×2 real matrix that is unitarily diagonalizable has one of the following forms: $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, for $a, b, d \in \mathbb{R}$.

Solution：

Section 9-3 problem 17.

10. (30 points) Prove or disprove the following statement:

(a) every unitarily diagonalizable matrix is Hermitian.

Solution：

9-3 problem 19(d): False!

From 9-3, Theorem 9.7.

A square matrix A is unitarily diagonalizable if and only if it is a normal matrix.

Therefore, just build a normal matrix which is not a Hermitian matrix as the counterexample.

(b) If U is unitary, then $(\bar{U})^{-1} = U^T$.

Solution：

9-2 problem 33(e): True!

(c) every unitary matrix is normal.

Solution：

9-2 problem 43(b): True!

(d) If $A^* = -A$, then A is normal.

Solution：

9-2 problem 43(c): True!

(e) $\det(C_{BB'}) = 1$ if and only if $B = B'$.

Solution：

From 7-1 problem 23 (h) : False!!

(f) If $C_{B,B'}$ is an orthogonal matrix and B is an orthonormal basis, then B' is an orthonormal basis.

Solution：

7-1 problem 23(e): True!

11. (10 points) Please give a $n \times n$ matrix (不需化簡，但需要理由) such that

(a) is diagonalizable but NOT a normal matrix.

Solution：

9-3

(b) is diagonalizable but NOT unitarily diagonalizable.

Solution：

9-3

(c) is unitarily diagonalizable matrix but NOT Hermitian.

Solution：

9-3

(d) all eigenvalues of algebraic multiplicity 1 but NOT unitarily diagonalizable.

Solution：

9-3

(e) two diagonalizable matrices having the same eigenvectors but NOT similar.

Solution：

7-2

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	10	10	10	10	10	10	10	30	10	130
Score:												