應數一線性代數 2023 秋, 第一次期中考 解答

學號: ______________________, 姓名: ______________________

本次考試共有 9 頁 (包含封面)，有 14 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
宏: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________

1. (10 points) (a) Find the inverse of the matrix A, if it exists, and (b) express the inverse matrix as a product of elementary matrices. $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

   Answer: (a) $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & 1 \end{bmatrix}$ (b) $A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ **(答案不唯一)**

2. (10 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T([1,0,0]) = [2,4,0], T([1,1,0]) = [3,4,3]$, and $T([1,2,3]) = [10,19,6]$. Find $T([4,-3,1]) =$ $[7,\ 21,\ -9]$

   Solution：

   $T([0,1,0]) = T([1,1,0] - [1,0,0]) = [1,0,3]$ and $T([0,0,1]) = \frac{1}{3}(T([1,2,3]) - T([1,0,0]) - 2T([0,1,0])) = [2,5,0]$. Let A is the standard matrix representation of T.

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 4 & 0 & 5 \\ 0 & 3 & 0 \end{bmatrix}$$

$$T([4,-3,1]) = (A \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix})^T = [7,\ 21,\ -9]$$

3. (5 points) Given $\vec{u} = [1, 3, -1]$, $\vec{v} = [-2, 4, 1]$ and $\vec{w} = [13, -11, -8]$.

   Is $\vec{w} \in sp(\vec{u}, \vec{v})$? ($\boxed{Yes}$ / No) .

   If so, write $\vec{w}$ in the linear combination of $\vec{v}$ and $\vec{u}$: $\vec{w} = 3\vec{u} - 5\vec{u}$.

4. (5 points) Given two vectors $\vec{v} = [3, x, -1, 1]$ and $\vec{u} = [6, 2, -2, y]$. Find all $x, y \in \mathbb{R}$ so that

   (a) $\vec{v}, \vec{u}$ are parallel. $x = 1,\ y = 2$.

   (b) $\vec{v}, \vec{u}$ are perpendicular. $x \in \mathbb{R},\ \ y = -2x - 20$.

   Solution：

   (a) Since the first component of $\vec{v}$ is 4 and the first component of $\vec{u}$ is 8, we know that $\vec{v}, \vec{u}$ are parallel if $2\vec{v} = \vec{u}$. Thus $x = 1, y = 2$.

   (b) It is fact that $\vec{v}, \vec{u}$ perpendicular if $\vec{v} \cdot \vec{u} = 0$. Since $\vec{v} \cdot \vec{u} = 18 + 2x + 2 + y$, $\vec{v}, \vec{u}$ perpendicular if $20 + 2x + y = 0$.

5. (5 points) Suppose that T is a linear transformation with standard matrix representation A, and that A is a 15×9 matrix such that the nullspace of A has dimension 3.

   (a) What is the dimension of the range of T? **6** .

   (b) What is the dimension of the kernel of T? **3** .

   Solution：

   Since the nullity of A is equal to 3, the rank of A is equal to 9-3=6. Thus the dimension of the kernel of T is 3, and the dimension of the range of T is 6.

6. (10 points) Consider the given linear system:

$$\begin{cases} x_1 - 2x_2 \qquad + \ x_4 = 1 \\ x_1 \qquad - x_3 + 2x_4 = 0 \\ \quad - 2x_2 + x_3 + \ x_4 = -6 \end{cases}$$

(a) Write its associated augmented matrix. $\left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 1 \\ 1 & 0 & -1 & 2 & 0 \\ 0 & -2 & 1 & 1 & -6 \end{array}\right]$

(b) Reduce the matrix to its reduced row-echelon form (rref). $\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 7 \\ 0 & 1 & -1/2 & 0 & 5/4 \\ 0 & 0 & 0 & 1 & -7/2 \end{array}\right]$.

(c) Find the homogeneous solution of the linear system . $\left\{ r \begin{bmatrix} 1 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$.

(d) Find the general solution of the linear system . $\left\{ \begin{bmatrix} 7 \\ 5/4 \\ 0 \\ -7/2 \end{bmatrix} + r \begin{bmatrix} 1 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \middle| \, r \in \mathbb{R} \right\}$.

7. (10 points) Let a, b and c be scalar such that $abc \neq 0$. Prove that the plane $ax + by + cz = 0$ is a subspace of $\mathbb{R}^3$.

Solution：

1-6 #12

8. (10 points) Assume the the matrix A can be row reduces to H, please answer the following questions.

$$A = \begin{bmatrix} 3 & 1 & -3 & 8 & 3 & 7 \\ -2 & -1 & 1 & -5 & 0 & -3 \\ 1 & 3 & 7 & 0 & 9 & 13 \\ 2 & 5 & 11 & 1 & -1 & 6 \end{bmatrix}, H = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & 1 \\ 0 & 1 & 3 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) the **rank** of matrix A, is **3** .

(b) a basis for the **row space** of A is **[1, 0, −2, 3, 0, 1], [0, 1, 3, −1, 0, 1],[0, 0, 0, 1, 1]** .

(c) a basis for the **column space** of A is $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 9 \\ -1 \end{bmatrix}$.

(d) a basis for the **nullspace** of A is $\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

9. (10 points) Let $\vec{v}_1$ and $\vec{v}_2$ be two vectors in $\mathbb{R}^n$. Prove that $sp(\vec{v}_1, \vec{v}_1 + \vec{v}_2) = sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2)$.

Solution：

1.6 #45

Clearly, $\vec{v}_1 = \frac{1}{2}[(\vec{v}_1 - \vec{v}_2) + (\vec{v}_1 + \vec{v}_2)]$ and $\vec{v}_1 + \vec{v}_2 = 0 \cdot (\vec{v}_1 - \vec{v}_2) + 1 \cdot (\vec{v}_1 + \vec{v}_2)$. Hence $\vec{v}_1, \vec{v}_1 + \vec{v}_2 \in sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2)$ and therefore $sp(\vec{v}_1, \vec{v}_1 + \vec{v}_2) \subset sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2)$.

Also, $\vec{v}_1 + \vec{v}_2 = 0 \cdot \vec{v}_1 + 1 \cdot (\vec{v}_1 + \vec{v}_2)$ and $\vec{v}_1 - \vec{v}_2 = 1 \cdot \vec{v}_1 - 1 \cdot (\vec{v}_1 + \vec{v}_2)$. Hence $\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2 \in sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2)$ and therefore $sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2) \subset sp(\vec{v}_1, \vec{v}_1 + \vec{v}_2)$.

Thus $sp(\vec{v}_1 - \vec{v}_2, \vec{v}_1 + \vec{v}_2) = sp(\vec{v}_1, \vec{v}_1 + \vec{v}_2)$

10. (10 points) Prove that the given relation holds for all matrices for which the expressions are defined.

$$(AB)^T = B^T A^T$$

Solution：

1-3 #32

11. (15 points) Suppose the complete solution to the equation

$$A\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \quad \text{is} \quad \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) The dimension of the row space of $A =$ **1**

(b) What is the matrix A? Answer: $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$.

(c) Find all possible $\vec{b}$ so that $A\vec{x} = \vec{b}$ can be solved. Answer: $\vec{b} = r \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$**, for** $r \in \mathbb{R}$.

12. (5 points) Let A is an $n \times n$ matrix and $A\vec{x} = \vec{b}$ has solution for any $\vec{b} \in \mathbb{R}^n$, then the nullity of A is **0** .

13. (15 points) Let A is an $m \times n$ matrix of rank r. Suppose $A\vec{x} = \vec{b}$ has *no solution* for some $\vec{b} \in \mathbb{R}^n$ and has *infinity many solution* for some other $\vec{b} \in \mathbb{R}^n$, then:

   (a) Is it possible that the nullspace of A contains only the zero vector? (Yes / *No*) , and why?

   (b) Is it possible that the column space of A is all of $\mathbb{R}^m$? (Yes / *No*) , and why?

   (c) Can there be a vector $\vec{b} \in \mathbb{R}^n$ for which $A\vec{x} = \vec{b}$ has exactly one solution? (Yes / *No*) , and why?

14. (10 points) Circle each of the following True or False and then give a counterexample （反例） for the false statement.

1. True **False** For all positive integers m and n, the nullity of an $m \times n$ matrix might be any number from 0 to n.

   Solution：
   For any 3×8 matrix, it could never have the nullity equals 0.

2. True **False** The magnitude of $\vec{v} + \vec{w}$ must be at least as large as the magnitude of either $\vec{v}$ or $\vec{w}$ in $\mathbb{R}^n$.

   Solution：
   Let $\vec{v} = [1, 0]$, $\vec{w} = [-1, 0]$, then $\|\vec{v} + \vec{w}\| = 0 < 1 = \|\vec{v}\| = \|\vec{w}\|$

3. **True** False If T is a linear transformation, then $T(\vec{0}) = 0$.

   Solution：
   2-3 p.144, 我上課有證！

4. True **False** If $A^2 = I$, then $A = \pm I$.

   Solution：

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)^2 = I$$

5. True **False** Every function mapping $\mathbb{R}^n$ into $\mathbb{R}^m$ is a linear transformation.

   Solution：
   Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T([x, y]) = [\cos(x) + y, 3x, 6y]$. T is NOT a linear transformation since $T(2[\pi, 0]) \neq 2T([\pi, 0])$.

學號: ________________, 姓名: ________________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	Total
Points:	10	10	5	5	5	10	10	10	10	10	15	5	15	10	130
Score:															