應數一線性代數 2023 秋, 期末考 解答

學號: ____________________, 姓名: ____________________

本次考試共有 10 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
宏: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ____________________

1. (10 points) Find the coordinate vector of the given vector relative to the indicated ordered basis.
$\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ in M_2 relative to $\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right)$.

   Answer: the coordinate vector is **[0, 5, 1, 4]**

2. (10 points) Let $\vec{a} = \vec{i} - 3\vec{k}$, $\vec{b} = -\vec{i} + 4\vec{j}$, $\vec{c} = \vec{i} + 2\vec{j} + \vec{k}$.
Find $\vec{a} \cdot (\vec{b} \times \vec{c}) =$ **22**

   Solution：

   純計算，如果你想用 octave 幫你算的話，可以輸入以下兩行即可得到答案

   a=[1 0 -3], b=[-1 4 0], c=[1 2 1]

   dot(a, cross(b, c))

3. (10 points) Let $T: P_2 \to P_3$ be defined by $T(ax^2+bx+c) = (4a+b-c)x^3+(2a+2b)x^2+(6b+c)x+(2a+b+3c)$, the ordered basis for P_2 is $B = (x^2, x, 1)$ and the ordered basis for P_3 is $B' = (x^3, x^2, x, 1)$. Fine the matrix representation A of T relative to the ordered bases B and B'.

Answer: (a) $A = \underline{\begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}}$

(b) find $p(x)$ such that $T(p(x)) = 7x^3+4x^2+4x-3$. $p(x) = \underline{x^2+x-2}$

Solution：

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 6 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 1 & -1 & 7 \\ 2 & 2 & 0 & 4 \\ 0 & 6 & 1 & 4 \\ 2 & 1 & 3 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

4. (10 points) Find the determinant of the given matrix.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 2 & 4 \\ 2 & 3 & 1 & 4 & 2 & 4 \\ 4 & 6 & 0 & 8 & 2 & 4 \\ -1 & 1 & 0 & -1 & 3 & -5 \\ 0 & 0 & 0 & 0 & 5 & 7 \\ 1 & 2 & 0 & -1 & 2 & 5 \end{bmatrix}$$

Answer: $\det(A) =$ **160**

Solution：

這題在寫的時候要特別注意書寫的符號，如果從頭到尾都沒用 determinant 的符號，先扣兩分。

除了注意 0 多的行或列在哪之外，如果有注意到第一個 row 跟最後一個 row 長得幾乎一樣，那就算得更快了。

5. (10 points)

$$A = \begin{bmatrix} 5 & -2 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

The inverse of A = $\frac{-1}{15}\begin{bmatrix} -2 & -1 & -2 \\ 3 & -6 & 3 \\ 1 & -7 & 16 \end{bmatrix}$, and the adjoint matrix of A = $\begin{bmatrix} -2 & -1 & -2 \\ 3 & -6 & 3 \\ 1 & -7 & 16 \end{bmatrix}$

Solution：

$\det(A) = -15$.

答案可以不用化簡。

6. (10 points) Determine the set S_1 of all functions f such that $f(0) = 1$ is a subspace in the vector space F of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

Answer: Is S_1 a subspace of F? **NO**

Solution：

Let $f(x), g(x) \in S_1$, then $(f \oplus g)(0) = f(0) + g(0) = 1 + 1 = 2 \neq 1$. It is NOT closed under vector addition. Therefore, $f \oplus g \notin S$ and S_1 is NOT a subspace of F.

7. (10 points) Determinant whether the given 4 points lie in a plane in $\mathbb{R}^4$. If so, find its area. If not, find its volume.

$$A(2,1,1,1), B(3,1,-1,2), C(2,0,2,3), D(2,-1,2,0)$$

Answer:

✗ $ABCD$ are coplanar(共平面), and the area of the quadrilateral (四邊形) is **N/A** .

✓ $ABCD$ are NOT coplanar, and the volume of the tetrahedron(四面體) is $\frac{\sqrt{156}}{6}$.

Solution：

$\overrightarrow{AB} = [1,0,-2,1], \overrightarrow{AC} = [0,-1,1,2], \overrightarrow{AD} = [0,-2,1,-1]$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ -2 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}, \quad \det(M^T M) = \begin{vmatrix} 6 & 0 & -3 \\ 0 & 6 & 1 \\ -3 & 1 & 6 \end{vmatrix} = 156$$

So the points are not coplanar and the volume of the Parallelepiped (平行六面體) formed by coterminous (相鄰邊) edges $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ is $\sqrt{156}$.

The volume of a tetrahedron (四面體) $ABCD$ formed by coterminous (相鄰邊) edges $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ is

$$\frac{\text{volume of the Parallelepiped}}{6} = \frac{\sqrt{156}}{6}$$

8. (10 points) Consider the set $\{(x, y)|\ x + y = 0\} \in \mathbb{R}^2$, with the addition defined by $[x, y] \oplus [a, b] = [x + a, y + b]$, and with scalar multiplication defined by $r \otimes [x, y] = [ry, rx]$.

   a. Is this set a vector space? __**No!**__
      Hint: Show by verifying the closed under two operations, A1-A4 and S1-S4.

   b. If it is a vector space, then what is the requested vectors in this vector space?
      Hint: The zero vector may NOT be the vector $[0, 0]$.
      Answer: the zero vector is _____ , for any vectors [x,y], the -[x,y] is _____

Solution：

Let $S = \{(x, y)|\ x + y = 0\}$.

S4. for any vector $[x, y] \in S$, we have $1 \oplus [x, y] = [y, x] \neq [x, y]$.

Therefore, it is NOT a vector space.

9. (10 points) Consider the set $\{(x,y)|\ x+y=1\} \in \mathbb{R}^2$, with the addition defined by $[x,y] \oplus [a,b] = [x+a+1, y+b]$, and with scalar multiplication defined by $r \otimes [x,y] = [rx+r-1, ry]$.

   a. Is this set a vector space? __**No!**__
      Hint: Show by verifying the closed under two operations, A1-A4 and S1-S4.

   b. If it is a vector space, then what is the requested vectors in this vector space?
      Hint: The zero vector may NOT be the vector $[0,0]$.
      Answer: the zero vector is _____ , for any vectors [x,y], the -[x,y] is _____

Solution：

Let $S = \{(x,y)|\ x+y=1\}$.

A0. for any $[x,y],[a,b] \in S$, we have $[x,y] \oplus [a,b] = [x+a+1, y+b]$. However, $([x,y] \oplus [a,b]) \notin S$ since $(x+a+1)+(y+b) = (x+y)+(a+b)+1 = 1+1+1 = 3 \neq 1$.

Therefore, it is NOT a vector space.

10. (10 points) Determine the dimension of the given set S. Then reduce the given set to be a basis for $sp(S)$.

$$S = sp(1, 4x+5, 5x-4, x^2+2, x-2x^2) \text{ is a subspce in a vector space } P.$$

Answer: $\dim(S) =$ **3** .

A basis for S is $\{1, 4x+5, x^2+2\}$

Solution：

$$\begin{bmatrix} 0 & 0 & 0 & 1 & -2 \\ 0 & 4 & 5 & 0 & 1 \\ 1 & 5 & -4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -4 & 2 & 0 \\ 0 & 4 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -10.25 & 0 & 2.75 \\ 0 & 1 & 1.25 & 0 & 0.25 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

11. (10 points) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $\hat{T} : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations. Prove directly from its definition that $(\hat{T} \circ T) : \mathbb{R}^n \to \mathbb{R}^k$ is also a linear transformation.

Solution：

2-3 #31.

我上課有證過。

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	10	10	10	10	10	10	10	10	10	110
Score:												