

應數一線性代數 2024 春, 期中考 解答

學號: _____, 姓名: _____

本次考試共有 10 頁 (包含封面), 有 10 題。如有缺頁或漏題, 請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號, 忘記填寫扣十分!
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程, 閱卷人員會視情況給予部份分數。
沒有計算過程, 就算回答正確答案也不會得到滿分。
答卷請清楚乾淨, 儘可能標記或是框出最終答案。

高師大校訓: 誠敬宏遠

誠, 一生動念都是誠實端正的。 敬, 就是對知識的認真尊重。
宏, 開拓視界, 恢宏心胸。 遠, 任重致遠, 不畏艱難。

請尊重自己也尊重其他同學, 考試時請勿東張西望交頭接耳。

1. (10 points) Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Is A orthogonal diagonalizable? (Yes / No) .

why? No! the algebraic multiplicity of 2 is 2, but the geometric multiplicity is 1

Solution :

First, A is orthogonal diagonalizable = there exist D is diagonal matrix and C is an orthogonal matrix such that $D = C^{-1}AC$.

The eigenvalues and eigenvectors of A are $(2, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}), (3, \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix})$.

Follow 課本 5-2 Theorem 5.4.

\therefore the algebraic multiplicity of 2 is 2, but the geometric multiplicity is 1.

$\therefore A$ is NOT diagonalizable.

$\therefore A$ is NOT orthogonal diagonalizable.

2. (10 points) Let

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & -4 \\ 6 & -4 & 3 \end{bmatrix}$$

Is A orthogonal diagonalizable? (Yes / No) .

why? Yes! A is symmetric!

Solution :

Method 1: Follow 課本 6-3 Theorem 6.8

Method 2:

$$\det(A - \lambda I) = -(\lambda^3 - 4\lambda^2 - 53\lambda + 124)$$

Let $f(x) = x^3 - 4x^2 - 53x + 124$, $f(0) = 124 > 0$, $f(6) = -122 < 0$.

By 高中堪根定理, $f(x)$ 有三相異實根分別在 $(-\infty, 0)$, $(0, 6)$, $(6, \infty)$ 的區間內。

定理 5.3 可知 A is diagonalizable.

定理 6.7 可知 symmtrix matrix 中, 相異的 eigenvalue 所對應的 eigenvector 是 orthogonal , 所以 A 是 orthogonal diagonalizable.

3. (10 points) Solve the system
$$\begin{cases} x'_1 = 4x_1 - 2x_2 + x_3 \\ x'_2 = -2x_1 + 3x_2 - 2x_3 \\ x'_3 = x_1 - 2x_2 + 4x_3 \end{cases}$$

Answer:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = C \begin{bmatrix} k_1 e^t \\ k_2 e^{3t} \\ k_3 e^{7t} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 e^t \\ k_2 e^{3t} \\ k_3 e^{7t} \end{bmatrix} = \begin{bmatrix} k_1 e^t - k_2 e^{3t} + k_3 e^{7t} \\ 2k_1 e^t - k_3 e^{7t} \\ k_1 e^t + k_2 e^{3t} + k_3 e^{7t} \end{bmatrix} .$$

Solution :

Follow 課本 5-3 example 3

Follow 109-2 midterm problem 1.

$$C = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

4. (15 points) Use Gram-Schmidt process to find an orthonormal basis for the subspace W of \mathbb{R}^4 spanned by $[1, 0, 1, 0]$, $[1, 1, -1, 0]$, $[1, 1, 0, 1]$ and then use it to find the QR-factorization of A , where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{42}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{42}} \\ 0 & 0 & \frac{-6}{\sqrt{42}} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{7}}{\sqrt{6}} \end{bmatrix},$$

an orthonormal basis of $W = \frac{1}{\sqrt{2}}[1, 0, 1, 0], \frac{1}{\sqrt{3}}[1, 1, -1, 0], \frac{1}{\sqrt{42}}[-1, 2, 1, 6],$

Solution :

Follow 課本 6-2 example 5.

Follow 111-2 quiz 6.

Follow 109-2 midterm problem 6.

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{42}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{42}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{42}} \\ 0 & 0 & \frac{6}{\sqrt{42}} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{3} & \frac{2}{\sqrt{3}} \\ 0 & 0 & \frac{\sqrt{7}}{\sqrt{6}} \end{bmatrix}$$

5. (10 points) Find the formula for the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that reflects in the line $3x - 7y = 0$.

Answer: $T([x, y]) = \underline{\frac{1}{58}[40x + 42y, 42x + 40y]}$.

Solution :

Follow 課本 5-2 example 2

Follow 109-2 midterm problem 2.

$$C = \begin{bmatrix} -3 & 7 \\ 7 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let A is the s.m.r. of T

$$A = CDC^{-1} = \frac{1}{-58} \begin{bmatrix} -40 & -42 \\ -42 & 40 \end{bmatrix}$$

6. (10 points) Find the projection of $[2, 4, 1]$ on the plane $P : 2x - y - 2z = 0$

Answer: the projection = $\frac{1}{9}[22, 34, 5]$, and the $P^\perp = \text{sp}\left(\begin{bmatrix} 2 & -1 & -2 \end{bmatrix}\right)$.

Solution :

Let $\vec{n} = \begin{bmatrix} 2 & -1 & -2 \end{bmatrix}$, then $P^\perp = \text{sp}(\vec{n})$.

Method 1

$$\vec{b}_{P^\perp} = \frac{\vec{b} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{-2}{9} [2, -1, -2] \Rightarrow \vec{b}_P = \vec{b} - \vec{b}_{P^\perp} = \frac{1}{9} [22, 34, 5]$$

Method 2 (i)

Since P is dim 2 subspace in \mathbb{R}^3 , we know $P = \text{sp}(\vec{a}_1, \vec{a}_2)$ for any $\vec{a}_1, \vec{a}_2 \in P$ and \vec{a}_1, \vec{a}_2 are not

parallel. Pick $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Let $A = [\vec{a}_1 \ \vec{a}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$. Then the projection is

$$\vec{b}_P = A(A^T A)^{-1} A^T \vec{b} = \frac{1}{9} [22, 34, 5]^T$$

Method 2 (ii)

Since $\vec{a}_1, \vec{a}_2, \vec{n}$ is a basis for \mathbb{R}^3 , and

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & 0 & -1 & 4 \\ 0 & 1 & -2 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 17/9 \\ 0 & 1 & 0 & 5/9 \\ 0 & 1 & 1 & -2/9 \end{array} \right] \Rightarrow \vec{b} = \frac{17}{9} \vec{a}_1 + \frac{5}{9} \vec{a}_2 + \frac{-2}{9} \vec{n}$$

$$\vec{b}_P = \frac{17}{9} \vec{a}_1 + \frac{5}{9} \vec{a}_2 = \frac{1}{9} [22, 34, 5], \quad \vec{b}_{P^\perp} = \frac{-2}{9} \vec{n} = \frac{-2}{9} [2, -1, -2]$$

Method 3

Pick $\vec{v}_1 = [1, 0, 1]$ from P , and let $\vec{v}_2 = \vec{v}_1 \times \vec{n} = [1, 4, -1]$. Find $\vec{q}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = \frac{1}{\sqrt{2}} [1, 0, 1]$,

$\vec{q}_2 = \frac{\vec{v}_2}{|\vec{v}_2|} = \frac{1}{\sqrt{18}} [1, 4, -1]$. Let $Q = [\vec{q}_1^T \ \vec{q}_2^T] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} \\ 0 & \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{18}} \end{bmatrix}$. Then the projection is

$$\vec{b}_P = Q Q^T \vec{b} = \frac{1}{9} \begin{bmatrix} 22 \\ 34 \\ 5 \end{bmatrix}$$

7. (10 points) Show that orthogonal matrices preserve the dot product of vectors. (i.e. $(A\vec{x} \cdot A\vec{y} = \vec{x} \cdot \vec{y})$.)

Solution :

Theorem 6.6 (1).

8. (10 points) Let A is an $n \times n$ invertible matrix and if λ is an eigenvalue of A with \vec{v} as a corresponding eigenvector. Prove that (a) $\lambda \neq 0$ and (b) $1/\lambda$ is an eigenvalue of A^{-1} with \vec{v} as a corresponding eigenvector.

Solution :

Section 5-1 # 28, 我上課有證過 and Quiz 1 ◦

9. (15 points) Circle True or False and then prove (證明) or disprove (反駁) it. Read each statement in original Greek before answering. *** 只圈對錯，沒有論述一律不給分 ***

- (a) True **False** Every $n \times k$ matrix A has a factorization $A = QR$, where the column vectors of Q form an orthonormal set and R is an invertible $k \times k$ matrix.

Solution :

Section 6-2, problem 25g.

- (b) True **False** Every vector in a vector space V is an eigenvector of the identity transformation of V into V .

Solution :

Section 5-1, problem 23i

- (c) **True** False Given W is a subspace of \mathbb{R}^n . If a vector \vec{v} belongs to both W and W^\perp , then $\vec{v} = \vec{0}$.

Solution :

上課證過

