應數一線性代數 2024 春, 期末考 解答

學號: ________________________________, 姓名: ________________________________

本次考試共有 10 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。　**敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。　**遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Express $\frac{z}{w}$ in the form $a+bi$, where $a, b \in \mathbb{R}$, if

$$z = -1 + i, \quad w = 5 + 4i$$

Answer: $\frac{z}{w} = \underline{\dfrac{-1+9i}{41}}$.

Solution：

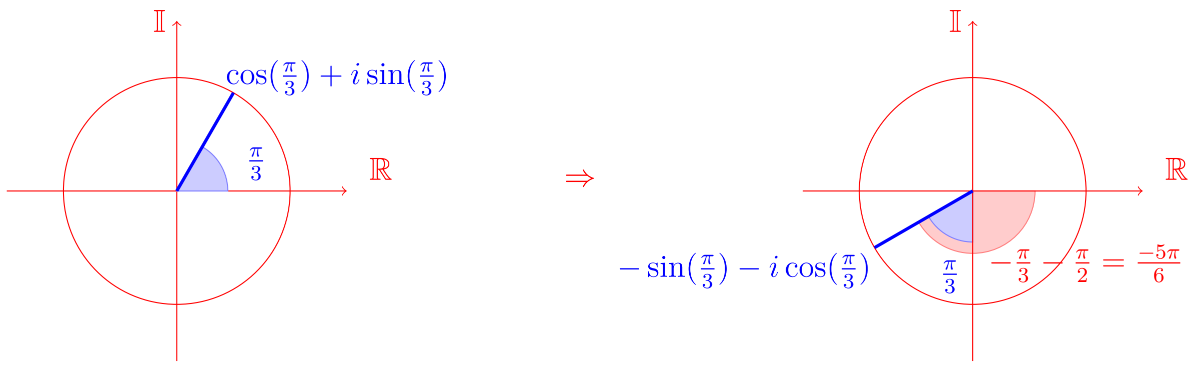
From 9-1. $$\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$$

2. (10 points) Find the five fifth roots of $-\sin(60°) - i\cos(60°)$. (need not simplify)

Answer: $\left(\cos\left(\frac{-\pi}{6} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{-\pi}{6} + \frac{2k\pi}{5}\right)\right)$, **for** $k = 0, 1, 2, 3, 4$, **(答案不唯一)**

Solution：

From 9-1. $-\sin(60°) - i\cos(30°) = -\sin(\frac{\pi}{3}) - i\cos(\frac{\pi}{3}) = \cos(\frac{-5\pi}{6}) + i\sin(\frac{-5\pi}{6})$.



$$\left(\cos\left(\frac{\frac{-5\pi}{6} + 2k\pi}{5}\right) + i\sin\left(\frac{\frac{-5\pi}{6} + 2k\pi}{5}\right)\right), \text{ for } k = 0, 1, 2, 3, 4$$

3. (10 points) Find an nonzero vector perpendicular to both $[i, 0, 1-i]$ and $[1+i, 1-i, 1]$ in $\mathbb{C}^3$.

Answer: $[-2i,\ 2+i,\ 1-i]$, **(答案不唯一)**

Solution：

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \bar{i} & \bar{0} & \overline{1-i} \\ \overline{1+i} & \overline{1-i} & \bar{1} \end{vmatrix} = [-2i,\ 2+i,\ 1-i]$$

4. (10 points) (1) Find the projection matrix P that project vectors in $\mathbb{R}^3$ on $W = sp([-1,0,1],[1,1,-1])$.

(2) Given $\vec{b} = [2,7,1]$, please find the projection $\vec{b}_W$.

(3) If $\vec{b}_W = \alpha[-1,0,1] + \beta[1,1,-1]$, find α, β.

Answer: $P =$ $\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $\vec{b}_W =$ $\frac{1}{2}[1,14,-1]$, $\alpha =$ **13/2**, $\beta =$ **7**.

Solution：

From 6-4.

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}, \; P = A(A^TA)^{-1}A^T$$

$$\vec{b}_W = P\vec{b}$$

5. (10 points) Find the least-square solution of the below system.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

Answer: The least-square solution = $\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$.

Solution：

From 6-5.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \;\Rightarrow\; (A^T A)\vec{x} = A^T\vec{b}$$

$$\Rightarrow\; \vec{x} = (A^T A)^{-1} A^T \vec{b} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

6. (10 points) Let V be a vector space with ordered bases $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $B' = \{\vec{b}'_1, \vec{b}'_2, \vec{b}'_3\}$. If

$$C_{B,B'} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}, \text{ and } \vec{v} = 3\vec{b}_1 - 2\vec{b}_2 + \vec{b}_3$$

Find the coordinate vector $\vec{v}_{B'} = \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix}$

Solution：

From 7-1.

$$\vec{v} = 3\vec{b}_1 - 2\vec{b}_2 + \vec{b}_3 \;\Rightarrow\; \vec{v}_B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{v}_{B'} = C_{B,B'}\vec{v}_B = \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix}$$

7. (10 points) Find the matrix representations $R_{B,B}$, $R_{B',B'}$ and an invertible C such that $R_{B',B'} = C^{-1}R_{B,B}C$ for linear transformation $T : P_2 \to P_2$ defined by $T(p(x)) = \frac{d}{dx}p(x+1)$, $B = (x^2, x, 1)$, $B' = (x^2-1, x-3, 2)$.

$C_{B,B'}=$ $\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, $C_{B',B}=$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$, $R_{B',B'}=$ $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 4 & 1/2 & 0 \end{bmatrix}$ and

$R_{B,B}=$ $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$.

Is C=$C_{B,B'}$ or $C_{B',B}$? $C_{B'B}$.

Solution：

From 7-2

$$T(x^2) = \frac{d}{dx}(x+1)^2 = 2x+2,\ T(x) = \frac{d}{dx}(x+1) = 1,\ T(1) = \frac{d}{dx}1 = 0$$

Thus

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = R_E\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

We have

$$R_{B,B} = R_B = R_E = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

By $C_{B',B} = M_B^{-1}M_{B'} = M_E^{-1}M_{B'} = I^{-1}M_{B'} = M_{B'}$,

$$C = C_{B',B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}$$

$$C_{B,B'} = C_{B',B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

Since

$$R_{B'} = R_{B',B'} = C_{B,B'}R_BC_{B',B}$$

$$R_{B'} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 4 & 1/2 & 0 \end{bmatrix}$$

8. (10 points) Find an unitary matrix U and a diagonal matrix D such that $D = U^{-1}AU$. Also find where

$$A = \begin{bmatrix} 2 & 0 & -1+i \\ 0 & -2 & 0 \\ -1-i & 0 & 1 \end{bmatrix}$$

Answer: D = $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, U = $\begin{bmatrix} 0 & \frac{1-i}{\sqrt{6}} & \frac{-1+i}{\sqrt{3}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

Solution：

From 9-3.

Since A is Hermitian matrix, it is unitarily diagonalizable.

It is easy to find $(-2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$, $(0, \begin{bmatrix} 1-i \\ 0 \\ 2 \end{bmatrix})$, $(3, \begin{bmatrix} -1+i \\ 0 \\ 1 \end{bmatrix})$ are three eigenvectors and its corresponding eigenvalues.

1. We also notice that $\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1-i \\ 0 \\ 2 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1+i \\ 0 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -1+i \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1-i \\ 0 \\ 2 \end{bmatrix} \right\rangle = 0$

or 2. Since A is Hermitian matrix who has three different eigenvalues, the three eigenvectors must be orthogonal to each other.

$$U = \begin{bmatrix} 0 & \frac{1-i}{\sqrt{6}} & \frac{-1+i}{\sqrt{3}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

9. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} 2 & 0 & 5 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Answer: Jordan canonical form = $J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$,

Jordan basis = $\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \vec{b}_4 & \vec{b}_5 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, **(不唯一)**

Solution：

From 9-4 and quiz 16.

$$rref(A-2I) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ null(A-2I) = sp(\vec{e}_1, \vec{e}_2, \vec{e}_4)$$

$$rref((A-2I)^2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, null((A-2I)^2) = sp(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4)$$

$$rref((A-2I)^2) = O, null((A-2I)^2) = \mathbb{C}^5$$

$$\begin{aligned} (A-2I) : \vec{b}_3 \to \vec{b}_2 \to \vec{b}_1 &\to \vec{0} \\ \vec{b}_4 &\to \vec{0} \\ \vec{b}_5 &\to \vec{0} \end{aligned}$$

Notice that $(\vec{b}_4, \vec{b}_5)$ can be $(\vec{e}_1, \vec{e}_4)$, $(\vec{e}_2, \vec{e}_4)$ or some other possibility, but not $(\vec{e}_1, \vec{e}_2)$.

10. (20 points) Match each matrix with its corresponding properties. Note that each matrix can have multiple properties, and some properties may apply to more than one matrix.

Properties: (a) diagonalizable (b) orthogonal diagonalizable (c) unitarily diagonalizable (d) symmetric (e) hermitian (f) normal (g) has reduced row-echelon form (h) has jordan canonical form

(i) $\begin{bmatrix} 2 & 3 & 0 & 1 & -1 \\ 3 & 0 & -2 & 5 & 1 \end{bmatrix}$. Answer: **(i) is a matrix: g**

(ii) $\begin{bmatrix} 5 & -1 & -2 \\ 1 & 3 & -2 \\ -1 & -1 & 4 \end{bmatrix}$. Answer: $\det(ii) = (2-\lambda)(4-\lambda)(6-\lambda)$**, not symmetric, not normal: agh**

(iii) $\begin{bmatrix} 1 & 1+i & 0 \\ 1-i & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Answer: **(iii) is a hermitian and not symmetric: acefgh**

(iv) $\begin{bmatrix} 1 & 2 & 6 \\ 2 & 0 & -4 \\ 6 & -4 & 3 \end{bmatrix}$. Answer: **(iv) is a real symmetric: abcdefgh**

Solution：

p.s. (iv) 跟期中考第二題一模一樣。

p.s.2. 考試的時候提醒過了，沒給理由沒分。

(a) **(Thm 5.4)** A is diagonalizable iff the a.m. = g.m. for each eigenvalue of A.

(b) **(Def in p.354)** A is orthogonal diagonalizable: 1. A is diagonalizable. 2. needs to check the eigenspaces are orthogonal.

  (1) **(6.3 #24)** If A is orthogonal diagonalizable then A is symmetric.

(c) **(Thm 9.7)** A is unitarily diagonalizable iff A is normal.

(d) **(Def 1.11)** A is symmetric if $A^T = A$.

  (1) **(Thm 6.8)** If A is real symmetric then A is orthogonal diagonalizable.

  (2) **(9.3 #19(h))** If A is real symmetric then A is normal.

(e) **(Def 9.4)** A is hermitian if $A^* = A$.

  (1) **(9.2 #43(a))** If A hermitian then A is normal.

(f) **(Def 9.5)** A is normal if $AA^* = A^*A$.

(g) **(Def in p.63)** A has reduced row-echelon form if A is a matrix.

(h) **(Thm 9.9)** A has jordan canonical form if A is a square matrix.

11. (10 points) Please give the $n \times n$ matrices (不需化簡，但需要理由) such that

(a) is diagonalizable but NOT unitarily diagonalizable.

Solution：

9-3, using **(Thm 9.7)** A is unitarily diagonalizable iff A is normal.

(b) is unitarily diagonalizable matrix but NOT Hermitian.

Solution：

9-3 #19(d).

(c) all eigenvalues of algebraic multiplicity 1 but NOT unitarily diagonalizable.

Solution：

9-3 #19(j).

(d) two diagonalizable matrices having the same eigenvectors but NOT similar.

Solution：

7-2 #23(i).

(e) Provide two ordered basis B and B' are not orthonormal bases, but $C_{B,B'}$ is an orthogonal matrix.

Solution：

7-1 #23(d).

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	10	10	10	10	10	10	10	10	10	20	10	120
Score:												