應數一線性代數 2024 秋, 第一次期中考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 9 頁 (包含封面)，有 12 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
宏: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________________

1. (5 points) Given $\vec{u} = [-1, 1, 2]$, $\vec{v} = [4, 2, -1]$ and $\vec{w} = [5, 7, 4]$.

   Is $\vec{w} \in sp(\vec{u}, \vec{v})$? ($\boxed{Yes}$ / No) .

   If so, write $\vec{w}$ in the linear combination of $\vec{v}$ and $\vec{u}$: $\vec{w} = 3\vec{u} + 2\vec{v}$.

2. (5 points) Given two vectors $\vec{v} = [x, 2, -1, 1]$ and $\vec{u} = [1, 6, -2, y]$. Find all $x, y \in \mathbb{R}$ so that

   (a) $\vec{v}, \vec{u}$ are parallel. **None** .

   (b) $\vec{v}, \vec{u}$ are perpendicular. $x \in \mathbb{R}$, $y = -x - 14$.

   Solution：

   (a) Since the second and third components of $\vec{v}$ is [2, -1] and the second and third components of $\vec{u}$ is [6, -2], we know that $\vec{v}, \vec{u}$ are never parallel.

   (b) It is fact that $\vec{v}, \vec{u}$ perpendicular if $\vec{v} \cdot \vec{u} = 0$. Since $\vec{v} \cdot \vec{u} = x + 12 + 2 + y$, $\vec{v}, \vec{u}$ perpendicular if $x + 14 + y = 0$.

3. (10 points) (a) Find the inverse of the matrix A, if it exists, and (b) express the inverse matrix as a product of elementary matrices. $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

Answer: (a) $A^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ (b) $A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ **(答案不唯一)**

4. (10 points) Describe all possible values for the unknowns x_i so that the matrix equation is valid.

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix}$$

Solution：

1-4, problem 35. $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

5. (10 points) Find all values of r for which A and B are commutes.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & r \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Answer: $r =$ **1**

6. (10 points) Let a, b and c be scalar such that $abc \neq 0$. Prove that the plane $ax + by + cz = 0$ is a subspace of $\mathbb{R}^3$.

Solution：

1-6 #12

7. (10 points) Assume the the matrix A can be row reduces to H, please answer the following questions.

$$A = \begin{bmatrix} 2 & 4 & 5 & 5 & 8 & 7 \\ -2 & -4 & -3 & 3 & 8 & 0 \\ 2 & 4 & 7 & 6 & 10 & -1 \\ 1 & 2 & 4 & 7 & 13 & 2 \end{bmatrix}, H = \begin{bmatrix} 1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) the **rank** of matrix A, is **4** .

(b) a basis for the **row space** of A is **[1, 2, 0, 0, -1, 0], [0, 0, 1, 0, 0, 0],[0, 0, 0, 1, 2, 0], [0, 0, 0, 0, 0, 1]** .

(c) a basis for the **column space** of A is $\begin{bmatrix} 2 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -1 \\ 2 \end{bmatrix}$.

(d) a basis for the **nullspace** of A is $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$.

8. (10 points) Use the previous question (前一題), let $\vec{a}_1 = [2, -2, 2, 1]$, $\vec{a}_2 = [4, -4, 4, 2]$, $\vec{a}_3 = [5, -3, 7, 4]$, $\vec{a}_4 = [5, 3, 6, 7]$, $\vec{a}_5 = [8, 8, 10, 13]$, $\vec{a}_6 = [7, 0, -1, 2]$

   (a) Is $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ a linear independent set? (Yes / *No*) .

   (b) Is $\{\vec{a}_1, \vec{a}_2\}$ a linear independent set?? (Yes / *No*) .

   (c) Is $\{\vec{a}_4, \vec{a}_5\}$ a linear independent set?? (*Yes* / No) .

   (d) Is $\{\vec{a}_1, \vec{a}_5, \vec{a}_6\}$ a linear independent set? (*Yes* / No) .

   (e) Is $\{\vec{a}_2, \vec{a}_4, \vec{a}_5\}$ a linear independent set? (Yes / *No*) .

   p.s. 記得每小題要分開給理由！!

9. (15 points) Suppose the complete solution to the equation

$$A\vec{x} = \begin{bmatrix}4\\2\\5\end{bmatrix} \quad \text{is} \quad \vec{x} = \begin{bmatrix}2\\0\\0\end{bmatrix} + r\begin{bmatrix}-2\\1\\0\end{bmatrix} + s\begin{bmatrix}0\\0\\1\end{bmatrix}$$

(a) The dimension of the row space of A = **1**

Solution：

Since $\vec{x}$ and $\begin{bmatrix}4\\2\\5\end{bmatrix}$ are both 3×1 matrices, we know that A is a 3×3 matrix.

$\vec{x}$ has two free variables, thus the nullity of A is 2. Therefore, the rank of A is 1.

(b) What is the matrix A? Answer: $A = \begin{bmatrix}2 & 4 & 0\\1 & 2 & 0\\0.5 & 5 & 0\end{bmatrix}$.

Solution：

Method 1:

$$\vec{x} = \begin{bmatrix}2-2r\\r\\s\end{bmatrix} \Rightarrow \left[\begin{array}{c|c} A & \begin{matrix}4\\2\\5\end{matrix}\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & 0 & 2\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right] \Rightarrow \left[\begin{array}{c|c} A & \begin{matrix}4\\2\\5\end{matrix}\end{array}\right] = \left[\begin{array}{ccc|c}2 & 4 & 0 & 4\\1 & 2 & 0 & 2\\0.5 & 5 & 0 & 5\end{array}\right]$$

Method 2:

1. $r = s = 0$:

$$A\vec{x} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}2\\0\\0\end{bmatrix} = \begin{bmatrix}4\\2\\5\end{bmatrix} \Rightarrow \begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix} = \begin{bmatrix}2\\1\\0.5\end{bmatrix}$$

2. $r = 1,\ s = 0$:

$$A\vec{x} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}4\\2\\5\end{bmatrix} \Rightarrow \begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix} = \begin{bmatrix}4\\2\\5\end{bmatrix}$$

3. $r = 1,\ s = 1$:

$$A\vec{x} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}4\\2\\5\end{bmatrix} \Rightarrow \begin{bmatrix}a_{12}+a_{13}\\a_{22}+a_{23}\\a_{32}+a_{33}\end{bmatrix} = \begin{bmatrix}4\\2\\5\end{bmatrix} \Rightarrow \begin{bmatrix}a_{13}\\a_{23}\\a_{33}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

(c) Find all possible $\vec{b}$ so that $A\vec{x} = \vec{b}$ can be solved. Answer: $\vec{b} = r\begin{bmatrix}4\\2\\5\end{bmatrix}$**, for** $r \in \mathbb{R}$.

Solution：

$$\vec{b} \in col(A)$$

10. (10 points) Prove that if A^3 is invertible, then A^2 is invertible.

Solution：

1-5, problem 23f.

11. (10 points) Let W_1 and W_2 be two subspace of $\mathbb{R}^n$. Prove that their intersection $W_1 \cap W_2$ is also a subspace.

Solution：

1-6, problem 47

Clearly $W_1 \cap W_2$ is nonempty; it contains 0.
Let $\vec{v}, \vec{w} \in (W_1 \cap W_2)$. Then $\vec{v}, \vec{w} \in W_1$ and $\vec{v}, \vec{w} \in W_2$, so $\vec{v} + \vec{w} \in W_1$ and $\vec{v} + \vec{w} \in W_2$ since W_1 and W_2 are subspaces.
Thus $\vec{v}+\vec{w} \in (W_1 \cap W_2)$. Similarly, $r\vec{v} \in W_1$ and $r\vec{v} \in W_2$. Since W_1 and W_2 are subspaces. Thus $r\vec{v} \in (W_1 \cap W_2)$.
Thus W_1 and W_2 are subspaces. Thus $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$

12. (10 points) Prove or disprove (反證) the following statement.

(a) Let $\vec{v}, \vec{w}$ be column vectors in $\mathbb{R}^n$ and let A be an $n \times n$ matrix. If $A\vec{v}$ and $A\vec{w}$ are linearly independent, then $\vec{v}$ and $\vec{w}$ are linearly independent

Solution：

It is true! 2-1, problem 36.

(b) Let $\vec{v}, \vec{w}$ be column vectors in $\mathbb{R}^n$ and let A be an $n \times n$ matrix. If $\vec{v}$ and $\vec{w}$ are linearly independent, then $A\vec{v}$ and $A\vec{w}$ are linearly independent

Solution：

It is false! Compare with 2-1, problem 34, the hypothesis missing the condition that A is invertible.

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	12	Total
Points:	5	5	10	10	10	10	10	10	15	10	10	10	115
Score:													