**應數一線性代數 2024 秋, 期末考 解答**

學號: ______________________________, 姓名: ______________________________

本次考試共有 9 頁 (包含封面)，有 12 題。如有缺頁或漏題，請立刻告知監考人員。

## 考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

**誠**: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
**宏**: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ______________________________

1. (10 points) Find the coordinate vector of the given vector relative to the indicated ordered basis.
$x + x^4$ in $P_2$ relative to $(1, (x+1), (x+1)^2, (x+1)^3, (x+1)^4)$.

Answer: the coordinate vector is **[0, -3, 6, -4, 1]**

**Solution：**

$$rref(\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 3 & 6 & 0 \\ 0 & 1 & 2 & 3 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]) = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

2. (10 points) Find the area of the parallelogram(平行四邊形) in $\mathbb{R}^3$ determined by the vectors $[2, -3, 5]$ and $[3, -2, 1]$

Answer: area = $\sqrt{243} = 9\sqrt{3}$

**Solution：**

$\boxed{By\ 4-1}$

Let $\vec{a} = [2, -3, 5]$ and $\vec{b} = [3, -2, 1]$, then the area is $\|\vec{a} \times \vec{b}\|$.

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 3 & -2 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -3 & 5 \\ -2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} = [7, 13, 5]$$

The length of $[7, 13, 5]$ is $\sqrt{243}$

$\boxed{By\ 4-4}$

Let

$$A = \begin{bmatrix} 2 & 3 \\ -3 & -2 \\ 5 & 1 \end{bmatrix}$$

$$\det(A^T A) = \det(\begin{bmatrix} 38 & 17 \\ 17 & 14 \end{bmatrix}) = 243$$

The volume of the 2-box is $\sqrt{243}$.

3. (10 points) Let $T : P_2 \to P_3$ be defined by $T(p(x)) = (x-2)p(x+1)$, the ordered basis for $P_2$ is $B = (x^2 - x, x^2 + x, 1)$ and the ordered basis for $P_3$ is $B' = (x^3, x^2, x, 1)$. Fine the standard matrix representation $A$ of $T$ relative to the ordered bases $B$ and $B'$.

Answer: (a) $A$ $\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ -2 & -4 & 1 \\ 0 & -4 & -2 \end{bmatrix}$

(b) Given $p(x)$ so that $p(x)_B = [1, 3, 2]$, find $p(x) =$ $4x^2 + 2x + 2$, and $T(p(x)) =$ $4x^3 + 2x^2 - 12x - 16$

**Solution：**

Similar with 3-4 exaqmple 9.

$$T(x^2 - x) = (x-2)[(x+1)^2 - (x+1)] = x^3 - x^2 - 2x, \qquad T(x^2 - x)_{B'} = [1, -1, -2, 0]$$

$$T(x^2 + x) = (x-2)[(x+1)^2 + (x+1)] = x^3 + x^2 - 4x - 4, \qquad T(x^2 + x)_{B'} = [1, 1, -4, -4]$$

$$T(1) = (x-2)[1] = (x-2), \qquad T(1)_{B'} = [0, 0, 1 - 2]$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ -2 & -4 & 1 \\ 0 & -4 & -2 \end{bmatrix}$$

$$T(p(x))_{B'} = Ap(x)_B = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ -2 & -4 & 1 \\ 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -12 \\ -16 \end{bmatrix}$$

$$p(x)_B = [1, 3, 2], \Rightarrow p(x) = 1(x^2 - x) + 3(x^2 + x) + 2(1) = 4x^2 + 2x + 2$$

$$T(p(x))_{B'} = \begin{bmatrix} 4 \\ 2 \\ -12 \\ -16 \end{bmatrix}, \Rightarrow T(p(x)) = 4x^3 + 2x^2 + (-12)x + (-16)1 = 4x^3 + 2x^2 - 12x - 16$$

4. (10 points) Find the determinant of the given matrix.

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 4 & 4 \\ 1 & 2 & 0 & -1 & 5 & 4 \\ 2 & 3 & 1 & 4 & 2 & 4 \\ 4 & 6 & 0 & 8 & 2 & 4 \\ -1 & 1 & 0 & -1 & 3 & -5 \\ 0 & 0 & 0 & 0 & 5 & 6 \end{bmatrix}$$

Answer: $\det(A) =$ **-192**

**Solution：**

這題在寫的時候要特別注意書寫的符號，如果從頭到尾都沒用 determinant 的符號，先扣兩分。

除了注意 0 多的行或列在哪之外，如果有注意到第一個 row 跟第二個 row 長得幾乎一樣，那就算得更快了。

5. (10 points)

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 3 & 2 & 1 \\ 1 & 5 & -1 \end{bmatrix}$$

The inverse of A = $\frac{1}{5}\begin{bmatrix} -7 & 3 & -4 \\ 4 & -1 & 3 \\ 13 & -2 & 6 \end{bmatrix}$ , and the adjoint matrix of A = $\begin{bmatrix} -7 & 3 & -4 \\ 4 & -1 & 3 \\ 13 & -2 & 6 \end{bmatrix}$

**Solution：**

$\boxed{4-3}$

$\det(A) = 5.$

答案可以不用化簡。

6. (10 points) Determine the set $S_1$ of all functions $f$ such that $f(0) = 0$ is a subspace in the vector space $F$ of all functions mapping $\mathbb{R}$ into $\mathbb{R}$.

Answer: Is $S_1$ a subspace of $F$? **Yes**

**Solution：**

$\boxed{3-2}$

Let $f(x), g(x) \in S_1$, then $(f \oplus g)(0) = f(0) + g(0) = 0 + 0 = 0$. Thus $(f \oplus g)(x) \in S_1$. It is closed under vector addition.

Let $r \in \mathbb{R}$, $(r \otimes f)(0) = r \times 0 = 0$. Thus $(r \otimes f)(x) \in S_1$. It is closed under scalar multiplication.

Thus $S_1$ is a subspace of $F$.

7. Consider the set $\mathbb{R}^2$, with the addition defined by $[x, y] \oplus [a, b] = [x + a + 2, y + b]$, and with scalar multiplication defined by $r \otimes [x, y] = [rx + r - 2, ry]$.

   a. Is this set a vector space? ( Yes / $\boxed{No}$ )
      *Hint:* Show by verifying the closed under two operations, A1-A4 and S1-S4.

   b. If the set is a vector space, then find the zero vector and the additive inverse (加法反元素) in this vector space. *Hint:* The zero vector may NOT be the vector $[0, 0]$.
      **Answer:** the zero vector is ____ , for any vectors [x,y], the -[x,y] is ____

**Solution：**

*Method* 1

Assume it is a vector space.

For any $\vec{v} \in \mathbb{R}^2$, $0 \otimes \vec{v}$ must be $\vec{0}$. (上課有證) Thus $\vec{0} = 0 \otimes [x, y] = [-2, 0]$.

For any $\vec{v} \in \mathbb{R}^2$, $(-1) \otimes \vec{v}$ must be $-\vec{v}$. (上課有證) Thus $-[x, y] = (-1) \otimes [x, y] = [-x - 3, -y]$.

However, $[x, y] \oplus (-[x, y]) = [x, y] \oplus [-x - 3, -y] = [x + (-x - 3) + 2, y + (-y)] = [-1, 0] \neq [-2, 0] = \vec{0}$. Contradiction!

Thus it is NOT a vector space.

*Method* 2

Check property S3: for $r, s \in \mathbb{R}$ and $\vec{v} = [x, y] \in \mathbb{R}^2$, $r \otimes (s \otimes \vec{v}) = (rs) \otimes \vec{v}$.

LHS: $r \otimes (s \otimes \vec{v}) = r \otimes (s \otimes [x, y]) = r \otimes [sx + s - 2, sy] = [r(sx + s - 2) + r - 2, rsy] = [rsx + rs - r - 2, rsy]$.

RHS: $(rs) \otimes \vec{v} = (rs) \otimes [x, y] = [(rs)x + (rs) - 2, (rs)y]$.

Since LHS $\neq$ RHS. It is NOT a vector space.

*Method* 3

Check property S4: for $\vec{v} = [x, y] \in \mathbb{R}^2$, $1 \otimes \vec{v} = \vec{v}$.

LHS: $1 \otimes \vec{v} = 1 \otimes [x, y] = [x + 1 - 2, y] = [x - 1, y]$.

RHS: $\vec{v} = [x, y]$.

Since LHS $\neq$ RHS. It is NOT a vector space.

8. (10 points) Determinant whether the given 4 points lie in a plane in $\mathbb{R}^4$. If so, find its area. If not, find its volume.

$$A(2,0,0,1), B(3,1,-1,2), C(2,0,2,3), D(2,-1,2,0)$$

Answer:

☒ $ABCD$ are coplanar(共平面), and the area of the quadrilateral (四邊形) is **N/A** .

☑ $ABCD$ are NOT coplanar, and the volume of the tetrahedron(四面體) is $\frac{\sqrt{48}}{6}$ .

**Solution：**

$\overrightarrow{AB} = [1,1,-1,1], \overrightarrow{AC} = [0,0,2,2], \overrightarrow{AD} = [0,-1,2,-1]$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 2 & 2 \\ 1 & 2 & -1 \end{bmatrix}, \quad \det(M^T M) = \begin{vmatrix} 4 & 0 & -4 \\ 0 & 8 & 2 \\ -4 & 2 & 6 \end{vmatrix} = 48$$

So the points are not coplanar and the volume of the Parallelepiped (平行六面體) formed by coterminous (相鄰邊) edges $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ is $\sqrt{48}$.

The volume of a tetrahedron (四面體) $ABCD$ formed by coterminous (相鄰邊) edges $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ is

$$\frac{\text{volume of the Parallelepiped}}{6} = \frac{\sqrt{48}}{6}$$

9. (10 points) Let $G = \{[x,y,z] \mid 0 \le x \le 3,\ 0 \le y \le 7,\ -2 \le z \le 5,\}$ Let $T: \mathbb{R}^3 \to \mathbb{R}^5$ be given by $T([x,y,z]) = [2x+3y, x-y, 2y+z, x+z, x-y-z]$. Find the volume of the image of G in $\mathbb{R}^5$ under the transformation $T$.

Answer: $147\sqrt{204}$

**Solution：**

Similar with example 7 in section 4-4.

Notice that $G$ is a cuboid with side length are 3, 7, 7, thus the volume of $G$ is $3 \times 7 \times 7 = 147$

By Theorem 4.9, we have the result should be $\sqrt{\det(A^T A)} \cdot V$ where $V$ is the volume of $G$, and $A$ is the standard matrix representation of $T$

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\det(A^T A) = \begin{vmatrix} 7 & 4 & 0 \\ 4 & 15 & 3 \\ 0 & 3 & 3 \end{vmatrix} = 204$$

10. (10 points) Determine the dimension of the given set $S$. Then reduce the given set to be a basis for $sp(S)$.

$$S = sp(x^2 - 2, x^2 + 1, 4x, 2x - 3) \text{ is a subspce in a vector space } P.$$

Answer: $\dim(S) =$ **3** .

A basis for $S$ is $\{x^2 - 2, x^2 + 1, 4x\}$

**Solution：**

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ -2 & 1 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0.5 \end{bmatrix}$$

11. (10 points) Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$. Show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

**Solution：**

Section 4-1 problem 59. 用定義驗證

本題特別要注意，驗證之前要先定義 $\vec{a}, \vec{b}, \vec{c}$ 的各 components 為何。

另外是向量的符號要用對。習慣上，$(a_1, a_2, a_3)$ 是指一個點，$[a_1, a_2, a_3]$ 才是一個向量。

12. (20 points) Prove or disprove (反證) the following statement.

(a) **True** False If $T$ and $\tilde{T}$ are different linear transformations mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, then we may have $T(\vec{e}_i) = \tilde{T}(\vec{e}_j)$ for some standard basis vector $\vec{e}_i$ of $\mathbb{R}^n$.

**Solution：**

2-3, problem 29h.

(b) **True** False Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $\hat{T} : \mathbb{R}^m \to \mathbb{R}^k$ be linear transformations. Prove directly from its definition that $(\hat{T} \circ T) : \mathbb{R}^n \to \mathbb{R}^k$ is also a linear transformation.

**Solution：**

2-3 #31.

(c) True **False** If $S$ is independent, each vector in $V$ can be expressed uniquely as a linear combination of vectors in $S$.

**Solution：**

3-2, 25(g)

(d) True **False** The determinant of a $3 \times 3$ matrix is zero if the points in $\mathbb{R}^3$ given by the rows of the matrix lie in a plane.

**Solution：**

4-1 19(e)

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 10 | 10 | 10 | 10 | 10 | 10 | 0 | 10 | 10 | 10 | 10 | 20 | 120 |
| Score: | | | | | | | | | | | | | |