應數一線性代數 2025 春, 期中考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 9 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，忘記填寫扣十分！
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。　**敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。　**遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Let

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 2 & 3 \\ -4 & 0 & -1 \end{bmatrix}$$

Find (if exists) an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$. Also, find the eigenvalues of A^{100}.

(1) The eigenvalue of A^{100} are $2^{100}, 3^{100}, 1$. (2) Is A diagonalizable?($\boxed{Yes}$ / No)

If A diagonalizable, C= $\begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, D= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$,

and A^{100}= $\begin{bmatrix} 3^{100} & 0 & 0 \\ 3^{100} - (-1)^{100} & 2^{100} & 2^{100} - (-1)^{100} \\ -3^{100} + (-1)^{100} & 0 & (-1)^{100} \end{bmatrix}$.

Solution：

$$C^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{100} = CD^{100}C^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 & 0 \\ 0 & 3^{100} & 0 \\ 0 & 0 & (-1)^{100} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{100} & 0 & 0 \\ 3^{100} - (-1)^{100} & 2^{100} & 2^{100} - (-1)^{100} \\ -3^{100} + (-1)^{100} & 0 & (-1)^{100} \end{bmatrix}$$

2. (10 points) Let

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 5 & 3 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

(a) Is A diagonalizable? (Yes / $\boxed{No}$) .

why? **eigenvalues are 3, 1, 1 and the geometric multiplicity of 1 is 1.**

(b) Is A orthogonal diagonalizable? (Yes / $\boxed{No}$) .

why? **以下兩個原因任意一個都可以：1. A is not diagonalizable, 2. A is not symmetric!**

Solution：

(a)

$$det(A - \lambda I) = (3 - \lambda)(1 - \lambda)^2$$

The eigenvalues are 3, 1, 1. The algebraic multiplicity of 3 is 1 and the algebraic multiplicity of 1 is 2.

$$A - I = \begin{bmatrix} -2 & 0 & -2 \\ 5 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \Rightarrow rank(A - I) = 2 \Rightarrow \text{nullity of } (A - I) = 2$$

Thus, the geometric multiplicity of 1 is 1. A is NOT diagonalizable.

(b) Follow 課本 6-3 Theorem 6.8

3. (15 points) Use Gram-Schmidt process to find an orthonormal basis for the subspace W of $\mathbb{R}^4$ spanned by the columns of A and then use it to find the QR-factorization of A, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Answer

Q= $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & \frac{-2}{\sqrt{10}} \\ 0 & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ 0 & \frac{-2}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} \sqrt{5} & 1 & -2 \\ 0 & 2 & 1 \\ \sqrt{5} & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$, R= $\begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{10}}{2} & \frac{-3}{\sqrt{10}} \\ 0 & 0 & \frac{-4}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} \sqrt{20} & \sqrt{5} & \sqrt{5} \\ 0 & 5 & -3 \\ 0 & 0 & -4 \end{bmatrix}$

Solution：

Follow 課本 6-2 example 5. R 除了課本上的方式，還可以用 $Q^T A$ 去求。

特別注意，QR 分解的 Q 一定是 orthogonal matrix，R 一定是上三角矩陣。

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & \frac{-2}{\sqrt{10}} \\ 0 & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ 0 & \frac{-2}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} \sqrt{5} & 1 & -2 \\ 0 & 2 & 1 \\ \sqrt{5} & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix}, R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{10}}{2} & \frac{-3}{\sqrt{10}} \\ 0 & 0 & \frac{-4}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} \sqrt{20} & \sqrt{5} & \sqrt{5} \\ 0 & 5 & -3 \\ 0 & 0 & -4 \end{bmatrix}$$

or

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{10}} & \frac{-2}{\sqrt{10}} \\ 0 & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \\ 0 & \frac{-2}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} \sqrt{5} & 1 & 2 \\ 0 & 2 & -1 \\ \sqrt{5} & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix}, R = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{10}}{2} & \frac{-3}{\sqrt{10}} \\ 0 & 0 & \frac{-4}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} \sqrt{20} & \sqrt{5} & \sqrt{5} \\ 0 & 5 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

4. (15 points) Let the sequence $a_0, a_1, \ldots$ given by $a_0 = 0, a_1 = 1$, and $a_k = a_{k-1} + 2a_{k-2}$ for $k \geq 2$. (1) Find the matrix A that can be used to generate this sequence. (2) Estimate(估計) a_k for large k.

Answer: $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$, $a_k = \frac{(-1)^{k+1}+2^k}{3}, \Rightarrow\ a_k \approx \frac{2^k}{3}, \Rightarrow\ \lim_{k\to\infty} a_k = \infty$

Solution：

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_k \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} a_{k-1} \\ a_{k-2} \end{bmatrix} = \ldots = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$$

Thus

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ has eigenvalues, eigenvectors as $\lambda_1 = -1, \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda_2 = 2, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

Method 1

Find C, D such that $A = CDC^{-1}$, where

$$C = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix},\ D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix},\ C^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix},$$

$$A^k = CD^kC^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & 2^k \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} (-1)^k + 2^{k+1} & 2(-1)^k + 2^{k+1} \\ (-1)^{k+1} + 2^k & (-2)(-1)^{k+1} + 2^k \end{bmatrix}$$

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = \frac{1}{3} \begin{bmatrix} (-1)^k + 2^{k+1} & 2(-1)^k + 2^{k+1} \\ (-1)^{k+1} + 2^k & (-2)(-1)^{k+1} + 2^k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} (-1)^{k+2} + 2^{k+1} \\ (-1)^{k+1} + 2^k \end{bmatrix}$$

Method 2

Find out that

$$\begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{k+1} \\ a_k \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^k \left(\frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \frac{1}{3}(-1)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{3}2^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}2^k \begin{bmatrix} (-1)^k + 2^{k+1} \\ (-1)^{k+1} + 2^k \end{bmatrix}$$

5. (10 points) Find the projection of [-1, 3, 2] on the subspace $W = sp([1,1,0],[1,0,1])$ in $\mathbb{R}^3$.

Answer:

1. the projection = $\underline{[1,1,0]}$. 2. the $W^\perp$ = $\underline{sp([1,-1,-1])}$.

Solution：

Method 1

Check section 6-1 example 3.

Method 2

Since we are in the $\mathbb{R}^3$ and $\dim(W) = 2$, then $\dim(W^\perp) = 3 - 2 = 1$.

$$\vec{n} = [1,1,0] \times [1,0,1] = [1,-1,-1] \ \Rightarrow W^\perp = sp([1,-1,-1]).$$

Let $\vec{v} = [-1,3,2]$, then

$$\vec{v}_{W^\perp} = \frac{\vec{v}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\,\vec{n} = [-2,2,2].$$

$$\vec{v}_W = \vec{v} - \vec{v}_{W^\perp} = [-1,3,2] - [-2,2,2] = [1,1,0].$$

6. (15 points) Let $\vec{v}$ be a vector in $\mathbb{R}^3$ with coordinate vector [3, 1, 6] relative to a ordered orthogonal basis ([2, 3, 6], [3, -6, 2], [6, 2, -3]) of $\mathbb{R}^3$. Find $\|\vec{v}\|$.

Answer: $\|\vec{v}\| =$ $\sqrt{46}$

Solution：

Method 1

我上課有說。Similar with 6-3 example 2.

Note that $\mathcal{A} = (\vec{a}_1 = [2, 3, 6], \vec{a}_2 = [3, -6, 2], \vec{a}_3 = [6, 2, -3])$ is an ordered orthogonal basis of $\mathbb{R}^3$ and $\|\vec{a}_1\| = \|\vec{a}_2\| = \|\vec{a}_3\| = 7$. Thus, $\mathcal{B} = (\vec{b}_1 = \frac{\vec{a}_1}{7}, \vec{b}_2 = \frac{\vec{a}_2}{7}, \vec{b}_3 = \frac{\vec{a}_3}{7})$ is an ordered orthonormal basis of $\mathbb{R}^3$.

Since

$$\vec{v} = 3\vec{a}_1 + 1\vec{a}_2 + 6\vec{a}_3 = 7(3\vec{b}_1 + 1\vec{b}_2 + 6\vec{b}_3),$$

$$\vec{v}_{\mathcal{A}} = [3, 1, 6],\ \ \vec{v}_{\mathcal{B}} = 7 \times [3, 1, 6],$$

By Theorem 6.6, we have $\|\vec{v}\| = \|\vec{v}_{\mathcal{B}}\| = 7 \times \sqrt{3^2 + 1^2 + 6^2} = 7\sqrt{46} = \sqrt{2254}$.

Method 2

By section 3-3:

$$\vec{v} = 3[2, 3, 6] + 1[3, -6, 2] + 6[6, 2, -3] = [45, 15, 2], \Rightarrow \|\vec{v}\| = \sqrt{2254}$$

7. (10 points) Let A is an $n \times n$ invertible matrix and if λ is an eigenvalue of A with $\overrightarrow{v}$ as a corresponding eigenvector. Prove that (a) $\lambda \neq 0$ and (b) $1/\lambda$ is an eigenvalue of A^{-1} with $\overrightarrow{v}$ as a corresponding eigenvector.

Solution：

Section 5-1 # 28。

(a) A is invertible, By Theorem 4.3, then $\det(A) \neq 0$.

Since $A\vec{v} = \lambda\vec{v}$, we have $(A - \lambda I)\vec{v} = \vec{0}$. Thus $(A - \lambda I)$ is singular and then $\det(A - \lambda I) = 0$.

If $\lambda = 0$,

$$0 = \det(A - \lambda I) = \det(A - 0I) = \det(A) \neq 0 \ \rightarrow\leftarrow$$

(b) A is invertible, then A^{-1} is exists.

$$\begin{aligned} A\vec{v} &= \lambda\vec{v} \\ \Rightarrow\ A^{-1}A\vec{v} &= A^{-1}\lambda\vec{v} \\ \Rightarrow\ \vec{v} &= \lambda A^{-1}\vec{v} \\ \Rightarrow\ \frac{1}{\lambda}\vec{v} &= A^{-1}\vec{v} \ \ (\text{Since } \lambda \neq 0) \end{aligned}$$

8. (15 points) Prove the statement if true; otherwise, modify it to make it true. (對的證明，錯的改正)*** **只圈對錯，沒有論述一律不給分** ***

(a) True **False** If λ is an eigenvalue of a matrix A, then λ is an eigenvalue of a matrix $A + cI$ for all scalars c.

Solution：

Section 5-1, problem 23g.

If λ is an eigenvalue of a matrix A, then ~~λ~~ $\lambda+cI$ is an eigenvalue of a matrix $A+cI$ for all scalars c.

(b) True **False** Every nonzero vector in $\mathbb{R}^n$ is in some orthonormal basis for $\mathbb{R}^n$.

Solution：

Section 6-2, problem 25e.

1. Every ~~*nonzero*~~ *unit* vector in $\mathbb{R}^n$ is in some orthonormal basis for $\mathbb{R}^n$.
2. Every nonzero vector in $\mathbb{R}^n$ is in some ~~*orthonormal*~~ *orthogonal* basis for $\mathbb{R}^n$.

(c) True **False** Given W is a subspace of $\mathbb{R}^n$. The intersection of W and $W^\perp$ is empty.

Solution：

Section 6-1, problem 23h. 見 112 exam 1 problem 9(c).

Given W is a subspace of $\mathbb{R}^n$. The intersection of W and $W^\perp$ is ~~*empty*~~ $\{\vec{0}\}$.

9. (10 points) Prove that similar square matrices have the same eigenvalues with the same algebraic multiplicities.

Solution：

Section 5-2, problem 18.

Let A and B are similar and $B = C^{-1}AC$. Then

$$\begin{aligned}\det(B - \lambda I) &= \det(C^{-1}AC - \lambda I) = \det(C^{-1}AC - C^{-1}(\lambda I)C) \\ &= \det(C^{-1}(A - \lambda I)C) = \det(C^{-1})\det(A - \lambda I)\det(C) \\ &= \det(A - \lambda I)\end{aligned}$$

Thus we know that A and B have the same characteristic polynomial. Therefore they have the same roots with the same multiplicities.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	15	15	10	15	10	15	10	110
Score:										