應數一線性代數 2025 春, 期末考

學號: ______________________________, 姓名: ______________________________

本次考試共有 10 頁 (包含封面)，有 9 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Express $(\sqrt{3}+i)^8$ in (1) the form $a+bi$ for a, b are real numbers, (2) the polar form.

   Answer: a=________________, b=________________, the polar form = ________________.

2. (10 points) Using the Gram-Schmidt process to transform the basis $\{[1,\ 1+i,\ 1-i], [1+i,\ 1-i,\ 1]\}$ into an orthogonal basis and then extend it as an orthogonal basis for $\mathbb{C}^3$.

   Answer: the found orthogonal basis for $\mathbb{C}^3$ is ________________________________

3. (10 points) (1) Find the projection matrix P that project vectors in $\mathbb{R}^3$ on W which is the plane $2x - y - 3z = 0$.

   (2) Given $\vec{b} = [2, 7, 1]$, please find the projection $\vec{b}_W$.

   Answer: $P =$ ________________________, $\vec{b}_W =$ ________________________

4. (10 points) Let V be a vector space with ordered bases $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $B' = \{\vec{b}'_1, \vec{b}'_2, \vec{b}'_3\}$. If

$$C_{B,B'} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}, \text{ and } \vec{v} = 3\vec{b}_1 - 2\vec{b}_2 + \vec{b}_3$$

Find the coordinate vector $\vec{v}_{B'} =$ ______________________________

5. (10 points) Find all the possible 2×2 real matrix that is unitarily diagonalizable.

6. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation and $B = ([-1, 1],\ [3, 3])$ and $B' = ([1, 1, 1],\ [2, 3, 1],\ [1, 2, 1])$ be ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Suppose that the matrix representation $R_{B,B'}$ of T is given by

$$R_{B,B'} = \begin{bmatrix} 1 & -2 \\ 4 & 2 \\ 2 & 0 \end{bmatrix}$$

Please express $T([1, 5])$ and $T([5, 1])$ as vectors in $\mathbb{R}^3$.

Answer: $T([1, 5])=$ __________________________, and $T([5, 1])=$ __________________________.

7. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} 5 & 0 & -1 & -1 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 1 & 2 & 7 & 2 & 1 \\ -1 & -2 & -2 & 3 & -1 \\ 0 & 1 & 1 & 1 & 5 \end{bmatrix}$$

(a) Jordan canonical form = ____________________, Jordan basis = ____________________

(b) Find the $\det(A^{50}) =$ ____.

Notice that

$$A-5I = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 \\ -1 & -2 & -2 & -2 & -1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, (A-5I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (A-5I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

8. (20 points) Match each matrix with its corresponding properties. Note that each matrix can have multiple properties, and some properties may apply to more than one matrix. (要寫理由)

Properties: (a) diagonalizable (b) orthogonal diagonalizable (c) unitarily diagonalizable (d) symmetric (e) hermitian (f) normal (g) has reduced row-echelon form (h) has jordan canonical form

(i) $\begin{bmatrix} 2 & 3 & 0 & 1 & -1 \\ 5 & 1 & -2 & 5 & 1 \end{bmatrix}$. Answer:____________________

(ii) $\begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix}$. Answer:____________________

(iii) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Answer:____________________

(iv) $\begin{bmatrix} 1 & 1+2i & 2-7i \\ 1-2i & 3i & 0 \\ 2+7i & 0 & -7 \end{bmatrix}$. Answer:____________________

(v) $\begin{bmatrix} 1 & 9 & -3 \\ 9 & 0 & -4 \\ -3 & -4 & 3 \end{bmatrix}$. Answer:____________________

(vi) $\begin{bmatrix} i & 4 \\ -4 & i \end{bmatrix}$. Answer:____________________

9. (10 points) Prove the following:
   (a) Show that every Hermitian matrix is normal.
   (b) Show that every unitary matrix is normal.
   (c) Show that, if $A^* = -A$, then A is normal.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	20	10	100
Score:										