應數一線性代數 2025 春, 期末考 解答

學號: ______________________________, 姓名: ______________________________

本次考試共有 8 頁 (包含封面)，有 9 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) Express $(\sqrt{3}+i)^8$ in (1) the form $a+bi$ for a, b are real numbers, (2) the polar form.

   Answer: a= -128 , b= $-128\sqrt{3}$, the polar form = $2^8(\cos\frac{-2\pi}{3}+i\sin\frac{-2\pi}{3})$.

   Solution：

$$
\begin{aligned}
(\sqrt{3}+i)^8 &= 2^8(\frac{\sqrt{3}}{2}+i\frac{1}{2})^8 = 2^8(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})^8 \\
&= 2^8(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}) = 2^8(-\frac{1}{2}+i\frac{-\sqrt{3}}{2}) \\
&= -128+i(-128\sqrt{3})
\end{aligned}
$$

   Notice that the principal argument θ should be $-\pi<\theta\leq\pi$.

$$
2^8(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}) = 2^8(\cos\frac{-2\pi}{3}+i\sin\frac{-2\pi}{3})
$$

2. (10 points) Using the Gram-Schmidt process to transform the basis $\{[1,\ 1+i,\ 1-i], [1+i,\ 1-i,\ 1]\}$ into an orthogonal basis and then extend it as an orthogonal basis for $\mathbb{C}^3$.

   Answer: the found orthogonal basis for $\mathbb{C}^3$ is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ **or** $\{\vec{v}_1, \vec{v}_2, \vec{v}_3'\}$

   Solution：

   Let $\vec{a}_1 = [1, 1+i, 1-i]$, $\vec{a}_2 = [1, 1+i, 1-i]$.

   Let

$$
\vec{v}_1 = \vec{a}_1 = [1,\ 1+i,\ 1-i].
$$

   Let

$$
\vec{v}_2 = \vec{a}_2 - \frac{\vec{v}_1\cdot\vec{a}_2}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 = \frac{1}{5}[3+5i,\ 3-7i,\ 3+2i]
$$

   Pick $\vec{a}_3 = [1, 0, 0]$, and then let

$$
\vec{v}_3 = \vec{a}_3 - \frac{\vec{v}_1\cdot\vec{a}_3}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2\cdot\vec{a}_3}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2 = \frac{1}{21}[10,\ 1+3i,\ -8+6i]
$$

   or use the crossing product

$$
\vec{v}_3' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \overline{1} & \overline{1+i} & \overline{1-i} \\ \overline{1+i} & \overline{1-i} & \overline{1} \end{vmatrix} = [1-3i,\ 1,\ 1+3i]
$$

   Notice that $\vec{v}_3$ and $\vec{v}_3'$ are parallel

$$
\vec{v}_3' = \frac{21\vec{v}_3}{1+3i}
$$

3. (10 points) (1) Find the projection matrix P that project vectors in $\mathbb{R}^3$ on W which is the plane $2x - y - 3z = 0$.

(2) Given $\vec{b} = [2, 7, 1]$, please find the projection $\vec{b}_W$.

Answer: $P = \frac{1}{14}\begin{bmatrix} 10 & 2 & 6 \\ 2 & 13 & -3 \\ 6 & -3 & 5 \end{bmatrix}$, $\vec{b}_W = \frac{1}{7}[20, 46, -2]$

Solution：

From 6-4.

Pick $\vec{a}_1 = [0, -3, 1]$ and $\vec{a}_2 = [1, 2, 0]$ are two linearly independent vectors in W.

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 2 \\ 1 & 0 \end{bmatrix},\ P = A(A^T A)^{-1} A^T$$

$$\vec{b}_W = P\vec{b}$$

4. (10 points) Let V be a vector space with ordered bases $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $B' = \{\vec{b}_1', \vec{b}_2', \vec{b}_3'\}$. If

$$C_{B,B'} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}, \text{ and } \vec{v} = 3\vec{b}_1 - 2\vec{b}_2 + \vec{b}_3$$

Find the coordinate vector $\vec{v}_{B'} = \underline{\begin{bmatrix} 0 \\ -4 \\ -4 \end{bmatrix}}$

Solution：

From 7-1.

$$\vec{v} = 3\vec{b}_1 - 2\vec{b}_2 + \vec{b}_3 \;\Rightarrow\; \vec{v}_B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{v}_{B'} = C_{B,B'}\vec{v}_B$$

5. (10 points) Find all the possible 2×2 real matrix that is unitarily diagonalizable.

Solution：

From 9-3 #17

Answer:

Every 2×2 real matrix A can written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since A is unitarily diagonalizable, A is normal, i.e. $A^*A = AA^*$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^*$$

$$\begin{bmatrix} a\bar{a} + c\bar{c} & \bar{a}b + \bar{c}d \\ a\bar{b} + c\bar{d} & b\bar{b} + d\bar{d} \end{bmatrix} = \begin{bmatrix} a\bar{a} + b\bar{b} & a\bar{c} + b\bar{d} \\ \bar{a}c + \bar{b}d & c\bar{c} + d\bar{d} \end{bmatrix}$$

Hence: (notice that a, b, c, d are real.)

(1). $a\bar{a} + c\bar{c} = a\bar{a} + b\bar{b} \Longrightarrow a^2 + c^2 = a^2 + b^2$

(2). $\bar{a}b + \bar{c}d = a\bar{c} + b\bar{d} \Longrightarrow ab + cd = ac + bd$

(3). $a\bar{b} + c\bar{d} = \bar{a}c + \bar{b}d \Longrightarrow ab + cd = ac + bd$

(4). $b\bar{b} + d\bar{d} = c\bar{c} + d\bar{d} \Longrightarrow b^2 + d^2 = c^2 + d^2$

by (1) and (4), we have $b = c$ or $b = -c$. And (2)(3) holds for for both cases.

Thus $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ -b & d \end{bmatrix}$ for $a, b, d \in \mathbb{R}$ are the only answer.

6. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation and $B = ([-1,1],\ [3,3])$ and $B' = ([1,1,1],\ [2,3,1],\ [1,2,1])$ be ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Suppose that the matrix representation $R_{B,B'}$ of T is given by

$$R_{B,B'} = \begin{bmatrix} 1 & -2 \\ 4 & 2 \\ 2 & 0 \end{bmatrix}$$

Please express $T([1,5])$ and $T([5,1])$ as vectors in $\mathbb{R}^3$.

Answer: $T([1,5])$= **[24, 38, 14]** , and $T([5,1])$= **[-20,-30,-14]** .

Solution：

By the definition, for every $\vec{v} \in \mathbb{R}^2$, we have

$$(T(\vec{v}))_{B'} = R_{B,B'}\vec{v}_B$$

$$[1,5] = 2[-1,1] + [3,3] \ \Rightarrow\ [1,5]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(T([1,5]))_{B'} = R_{B,B'}[1,5]_B = \begin{bmatrix} 1 & -2 \\ 4 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 4 \end{bmatrix}$$

$$T([1,5]) = 0[1,1,1] + 10[2,3,1] + 4[1,2,1] = [24,38,14]$$

Similarly,

$$[5,1]_B = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$(T([5,1]))_{B'} = \begin{bmatrix} -4 \\ -6 \\ -4 \end{bmatrix} \ \Rightarrow\ T([5,1]) = [-20,-30,-14]$$

7. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} 5 & 0 & -1 & -1 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 1 & 2 & 7 & 2 & 1 \\ -1 & -2 & -2 & 3 & -1 \\ 0 & 1 & 1 & 1 & 5 \end{bmatrix}$$

(a) Jordan canonical form = $\underline{\ J\ }$, Jordan basis = **ordered column of C, (不唯一)**

(b) Find the $\det(A^{50}) = \underline{(5^5)^{50} = 5^{250}}$.

Notice that

$$A-5I = \begin{bmatrix} 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 \\ -1 & -2 & -2 & -2 & -1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, (A-5I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (A-5I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution：

$$rref(A-5I) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ null(A-5I) = sp(\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix})$$

$$rref((A-5I)^2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ null((A-5I)^2) = sp(\vec{e}_1, \vec{e}_3, \vec{e}_4, \vec{e}_5)$$

$$rref((A-5I)^3) = O,\ null((A-5I)^3) = sp(\vec{e}_1, \vec{e}_2, \vec{e}_4, \vec{e}_5)$$

$$\begin{aligned} (A-2I): \quad & \vec{b}_2 \to \vec{b}_1 \to \vec{0} \\ & \vec{b}_5 \to \vec{b}_4 \to \vec{b}_3 \to \vec{0} \end{aligned}$$

$$\text{Pick } \vec{b}_5 = \vec{e}_2,\ \vec{b}_4 = (A-5I)\vec{b}_5 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 1 \end{bmatrix},\ \vec{b}_3 = (A-5I)\vec{b}_4 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix},$$

$$\text{Pick } \vec{b}_2 = \vec{e}_4,\ \vec{b}_1 = (A-5I)\vec{b}_2 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 2 & 0 \\ -2 & 0 & -1 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, J = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$A = CJC^{-1}$$

$$\det(A)^{50} = \det(J)^{50} = (5^5)^{50} = 5^{250}$$

8. (20 points) Match each matrix with its corresponding properties. Note that each matrix can have multiple properties, and some properties may apply to more than one matrix. (要寫理由)

Properties: (a) diagonalizable (b) orthogonal diagonalizable (c) unitarily diagonalizable (d) symmetric (e) hermitian (f) normal (g) has reduced row-echelon form (h) has jordan canonical form

(i) $\begin{bmatrix} 2 & 3 & 0 & 1 & -1 \\ 5 & 1 & -2 & 5 & 1 \end{bmatrix}$. Answer: **g, it is a matrix**

(ii) $\begin{bmatrix} -3 & 5 & -20 \\ 2 & 0 & 8 \\ 2 & 1 & 7 \end{bmatrix}$. Answer: **agh, three distinct eigenvalue $\Rightarrow$ (a), not symmetric, not normal**

(iii) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Answer: **gh, it is in the Jordan form $\Rightarrow$ Not diagonalizable**

(iv) $\begin{bmatrix} 1 & 1+2i & 2-7i \\ 1-2i & 3i & 0 \\ 2+7i & 0 & -7 \end{bmatrix}$. Answer: **acefgh, it is a hermitian and not symmetric**

(v) $\begin{bmatrix} 1 & 9 & -3 \\ 9 & 0 & -4 \\ -3 & -4 & 3 \end{bmatrix}$. Answer: **abcdefgh, it is a real symmetric**

(vi) $\begin{bmatrix} i & 4 \\ -4 & i \end{bmatrix}$. Answer: **acfgh, it is normal but not hermitian, not symmetric**

Solution：

考試的時候提醒過了，沒給理由沒分。注意：選擇跟不選擇都要有理由！

(a) **(Thm 5.4)** A is diagonalizable iff the a.m. = g.m. for each eigenvalue of A.

(b) **(Def in p.354)** A is orthogonal diagonalizable: 1. A is diagonalizable. 2. needs to check the eigenspaces are orthogonal.

   (1) **(6.3 #24)** If A is orthogonal diagonalizable then A is symmetric.

(c) **(Thm 9.7)** A is unitarily diagonalizable iff A is normal.

(d) **(Def 1.11)** A is symmetric if $A^T = A$.

   (1) **(Thm 6.8)** If A is real symmetric then A is orthogonal diagonalizable.

   (2) **(9.3 #19(h))** If A is real symmetric then A is normal.

(e) **(Def 9.4)** A is hermitian if $A^* = A$.

   (1) **(9.2 #43(a))** If A hermitian then A is normal.

(f) **(Def 9.5)** A is normal if $AA^* = A^*A$.

(g) **(Def in p.63)** A has reduced row-echelon form if A is a matrix.

(h) **(Thm 9.9)** A has jordan canonical form if A is a square matrix.

9. (10 points) Prove the following:
   (a) Show that every Hermitian matrix is normal.
   (b) Show that every unitary matrix is normal.
   (c) Show that, if $A^* = -A$, then A is normal.

Solution：

From 9-2 #43

Answer:

(a) Let H are Hermitian matrices, i.e. $H^* = H$. $HH^* = HH = H^*H$.

(b) Let U are unitary matrices, i.e. $U^*U = I$, i.e. $U^{-1} = U^*$. $UU^* = I = U^*U$.

(c) If $A^* = -A$, $A^*A = (-A)A = -AA = A(-A) = AA^*$.

學號: ____________, 姓名: ____________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	20	10	100
Score:										