應數一線性代數 2025 秋, 第一次期中考 解答

學號: ____________________, 姓名: ____________________

本次考試共有 10 頁 (包含封面)，有 11 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠: 一生動念都是誠實端正的。 **敬**: 就是對知識的認真尊重。
宏: 開拓視界，恢宏心胸。 **遠**: 任重致遠，不畏艱難。

請簽名保證以下答題都是由你自己作答的，並沒有得到任何的外部幫助。

簽名: ____________________

1. (a) (4 points) If $A = [a_{i,j}]$ is a $m \times n$ matrix and $B = [b_{i,j}]$ is a $n \times r$ matrix. Define what it means for the matrix product $C = AB$.

Solution：

Definition 1.8 in Section 1.3.

(b) (4 points) Define what it means for two vectors $\vec{v}, \vec{u} \in \mathbb{R}^n$ are perpendicular ($\vec{v} \perp \vec{u}$).

Solution：

Definition 1.7 in Section 1.2.

(c) (4 points) Define what it means for a $n \times n$ matrix A is invertible.

Solution：

Definition 1.15 in Section 1.5.

(d) (4 points) Define what it means for a set of vectors to be linearly independent.

Solution：

Definition 2.1 in Section 2.1.

(e) (4 points) If W is a subspace of $\mathbb{R}^n$, define the dimension of W.

Solution：

Definition 2.2 in Section 2.1.

2. (10 points) (a) Find the inverse of the matrix A, if it exists, and (b) express the inverse matrix as a product of elementary matrices.

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 0 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

Answer: (a) $A^{-1} = \frac{1}{3}\begin{bmatrix} 4 & 6 & 7 \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{bmatrix}$

(b) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **(答案不唯一)**

3. (5 points) Classify $\vec{v} = [4, 1, 2, 1, 6]$ and $\vec{u} = [8, 2, 4, 2, -5]$ are parallel, perpendicular, or neither.

Answer: **neither**

4. (10 points) Let $W = \{[a_1, a_2, a_3, a_4, a_5] \in \mathbb{R}^5 \mid a_1 + a_2 + a_3 + a_4 + a_5 = 0\}$. Show that W is a subspace of $\mathbb{R}^5$ over $\mathbb{R}$.

Solution：

First, we know that W is a subset of $\mathbb{R}^5$.

For any $\vec{u}, \vec{v} \in \mathbb{R}^5$, say $\vec{u} = [u_1, u_2, u_3, u_4, u_5], \vec{v} = [v_1, v_2, v_3, v_4, v_5]$.

If $\vec{u}, \vec{v} \in W$, we know that $u_1 + u_2 + u_3 + u_4 + u_5 = 0$ and $v_1 + v_2 + v_3 + v_4 + v_5 = 0$

For any scalar $r, s \in \mathbb{R}$.

$$\begin{aligned} r\vec{u} + s\vec{v} &= r[u_1, u_2, u_3, u_4, u_5] + s[v_1, v_2, v_3, v_4, v_5] \\ &= [ru_1, ru_2, ru_3, ru_4, ru_5] + [sv_1, sv_2, sv_3, sv_4, sv_5] \\ &= [ru_1 + sv_1, ru_2 + sv_2, ru_3 + sv_3, ru_4 + sv_4, ru_5 + sv_5] \end{aligned}$$

Since $(ru_1 + sv_1) + (ru_2 + sv_2) + (ru_3 + sv_3) + (ru_4 + sv_4) + (ru_5 + sv_5) = r(u_1 + u_2 + u_3 + u_4 + u_5) + s(v_1 + v_2 + v_3 + v_4 + v_5) = r(0) + s(0) = 0$, the $r\vec{u} + s\vec{v}$ is in W. Thus W is a subspace of $\mathbb{R}^5$

5. (10 points) Find all possible scalar s, if exist, such that $[1, 0, 1], [2, s, 3], [1, -s, 0]$ are linearly independent.

Solution：

problem 33 in Section 2.1. (我忘記改數字了)

6. (10 points) For the following linear system:

$$\begin{cases} x_1 + 2x_2 - 3x_3 = & 4 \\ 3x_1 - x_2 + 5x_3 = & 2 \\ 4x_1 + x_2 + ax_3 = & b \end{cases}$$

Find all possible (a, b) such that the above linear system

(a) has unique solutions: $(a, b) =$ $a \neq 2,\ b \in \mathbb{R}$.

(b) has no solutions: $(a, b) =$ $a = 2,\ b \neq 6$.

(c) has infinitely many solutions: $(a, b) =$ $(2, 6)$.

Solution：

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a-2 & b-6 \end{array}\right]$$

Using the Theorem 1.7 in Section 1.4.

7. (20 points) Assume the the matrix A is row equivalent to H, please answer the following questions.

$$A = \begin{bmatrix} 1 & 1 & 5 & 0 & -9 & 3 & -3 \\ 2 & 4 & 20 & 1 & -28 & 14 & -14 \\ 2 & 4 & 20 & 1 & -31 & 15 & -15 \\ 2 & 3 & 15 & 1 & -27 & 12 & -11 \\ 1 & 2 & 10 & 1 & -18 & 9 & -8 \end{bmatrix}, H = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & -10 & 16 \\ 0 & 6 & 30 & 0 & 0 & 10 & -16 \\ 0 & 0 & 0 & 6 & 0 & 8 & 4 \\ 0 & 0 & 0 & 0 & 6 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) the **rank** of matrix A, is __**4**__ .

(b) a basis for the **row space** of A is
$\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & -10 & 16 \end{bmatrix}, \begin{bmatrix} 0 & 6 & 30 & 0 & 0 & 10 & -16 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 6 & 0 & 8 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 6 & -2 & 2 \end{bmatrix},$
.

(c) a basis for the **column space** of A is $\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ -28 \\ -31 \\ -27 \\ -18 \end{bmatrix}$.

(d) a basis for the **nullspace** of A is $\begin{bmatrix} 0 \\ -30 \\ 6 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ -10 \\ 0 \\ -8 \\ 2 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -16 \\ 16 \\ 0 \\ -4 \\ -2 \\ 0 \\ 6 \end{bmatrix}$.

(e) Is [9, 28, 31, 27, 18] in sp([1, 2, 2, 2, 1], [1, 4, 4, 3, 2], [5, 20, 20, 15, 10], [0, 1, 1, 1, 1])? If so, find the coefficient of its linear combination. If not, explain why. Answer: __**No**__

(f) Is [3, 14, 15, 12, 9] in sp([1, 2, 2, 2, 1], [1, 4, 4, 3, 2], [5, 20, 20, 15, 10], [0, 1, 1, 1, 1], [9, 28, 31, 27, 18])? If so, find the coefficient of its linear combination. If not, explain why. Answer: __**Yes**__

Solution：

By Theorem 1.6 in Section 1.4, Since A and H are row equivalent, $A\vec{x} = \vec{0}$ and $H\vec{x} = \vec{0}$ has the same solution set. Let $\vec{h}_i$ be the i^{th} column of H and $\vec{a}_i$ be the i^{th} column of A .

(e) From H, the pivot columns are $1, 2, 4, 5$, and the free columns are $3, 6, 7$. Thus, column 5 is a pivot column, meaning $\vec{h}_5$ cannot be written as a linear combination of the previous columns. Hence $[9, 28, 31, 27, 18]^T = -\vec{a}_5$ is *not* in span$(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4)$.

(f) Column 6 is a free column in H, so it can be expressed as a linear combination of the pivot columns.

$$\vec{h}_6 = -\tfrac{5}{3}\vec{h}_1 + \tfrac{5}{3}\vec{h}_2 + 0\vec{h}_3 + \tfrac{4}{3}\vec{h}_4 - \tfrac{1}{3}\vec{h}_5. \text{ (表示不唯一)}$$

Therefore, $[3, 14, 15, 12, 9]^T = \vec{a}_6$ is in span$(\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5)$ and

$$\vec{a}_6 = -\tfrac{5}{3}\vec{a}_1 + \tfrac{5}{3}\vec{a}_2 + 0\vec{a}_3 + \tfrac{4}{3}\vec{a}_4 - \tfrac{1}{3}\vec{a}_5. \text{ (表示不唯一)}$$

(d) Similar with 111-1 exam 1 problem 8.

8. (10 points) Let A be an $m \times n$ matrix. Let $\vec{e}_j$ be the $n \times 1$ column vector whose j^{th} component is 1 and whose other components are 0.

   (a) Show that $A\vec{e}_j$ is the j^{th} column vector of A.

   Solution：

   Section 1.3 problem 41 (a). Use the definition of $A\vec{e}_j$ to check.

   (b) Prove that if $A\vec{x} = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$, then $A = O$, the zero matrix.

   Solution：

   Section 1.3 problem 41 (b)(i). By above part and use $\vec{x} = \vec{e}_j$ for $j = 1, 2, ..., n$.

9. (10 points) Let A and C be matrices such that the product AC is defined. Whether the column space of AC is contained in the column space of A or C? Explain your answer.

Answer: the column space of AC is contained in the column space of __A__ .

Solution：

Section 2.2 problem 14, 15.

1. Let A be $m \times n$ matrix. Every vector in the column space of AC is of the form $\vec{v} = (AC)\vec{x}$ for some $\vec{x} \in \mathbb{R}^n$. For every $\vec{x}$, $(C\vec{x}) \in \mathbb{R}^n$. Then $\vec{v} = A(C\vec{x})$ which is the vector in the column space of A. Thus $colspace(AC) \subseteq colspace(A)$.
2. Let A be $m \times n$ matrix, C be $n \times s$ matrix. Thus AC is $m \times s$ matrix.

   Since the column vectors of AC are belong to $\mathbb{R}^m$ and the column vectors of C are belong to $\mathbb{R}^n$, the column space of AC can not be contained in the column space of C.

10. (5 points) Let $\vec{v}, \vec{u}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$. Prove that

$$\vec{v} \cdot (\vec{u} + \vec{w}) = \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{w}$$

Solution：

Section 1.2, Theorem 1.3 (D.2).

Check by it's definition.

11. (5 points) Prove that $sp(\vec{v}_1, \vec{v}_2) = sp(\vec{v}_1 + \vec{v}_2, \vec{v}_1 - 2\vec{v}_2)$.

Solution：

Similar with Section 1.6, problem 45.

學號: ______________, 姓名: ______________, **以下由閱卷人員填寫**

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	20	10	5	10	10	10	20	10	10	5	5	115
Score:												