

應數一線性代數 2024 秋, 期末考 解答

學號: _____, 姓名: _____

本次考試共有 9 頁 (包含封面), 有 13 題。如有缺頁或漏題, 請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- 不可翻閱課本或筆記。
- 1-7 題為填空題。
- 8-13 題為計算證明題。請寫出計算過程, 閱卷人員會視情況給予部份分數。沒有計算過程, 就算回答正確答案也不會得到滿分。答卷請清楚乾淨, 儘可能標記或是框出最終答案。
- 書寫空間不夠時, 可利用試卷背面, 但須標記清楚。

高師大校訓: 誠敬宏遠

誠: 一生動念都是誠實端正的。 敬: 就是對知識的認真尊重。
宏: 開拓視界, 恢宏心胸。 遠: 任重致遠, 不畏艱難。

請簽名保證以下答題都是由你自己作答的, 並沒有得到任何的外部幫助。

簽名: _____

1. (10 points) Linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ satisfy $T([1, 3]) = [2, 2, a]$, and $T([2, 1]) = [3, b, 6]$. If T is NOT one-to-one, then $a + b =$ (1) 7

Solution :

Since $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is not one-to-one and the dimension of the domain is 2, the vectors $T([1, 3])$ and $T([2, 1])$ must be linearly dependent. This implies that the vectors $[2, 2, a]^T$ and $[3, b, 6]^T$ are proportional:

$$\frac{2}{3} = \frac{2}{b} = \frac{a}{6}$$

From $\frac{2}{3} = \frac{2}{b}$, we get $b = 3$. From $\frac{2}{3} = \frac{a}{6}$, we get $a = 4$. Thus, $a + b = 4 + 3 = 7$.

2. (10 points) Given B and the inverse matrix of B are below, then $a =$ (2) 0.8

$$B = \begin{bmatrix} 0 & -2 & 1 \\ 3 & 2 & 1 \\ 1 & 5 & -1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} * & * & * \\ a & * & * \\ * & * & * \end{bmatrix}$$

Solution :

Using the formula for the inverse matrix $B^{-1} = \frac{1}{\det(B)} \text{adj}(B)$, the element a at position $(2, 1)$ of B^{-1} is:

$$a = \frac{1}{\det(B)} (-1)^{1+2} \det(B_{12})$$

$$\det(B) = 5, \quad \det(B_{12}) = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4 \implies a = \frac{1}{5} (-1) (-4) = 0.8$$

3. (10 points) Suppose that T is a linear transformation with standard matrix representation A , and that A is a 9×15 matrix such that the nullspace of A has dimension 5.

(a) The dimension of the range of T is (3) 10. (b) The dimension of the kernel of T is (4) 5.

Solution :

By the Rank-Nullity Theorem: $\text{rank}(T) + \text{nullity}(T) = \dim(\text{Domain})$. Given $\dim(\text{Domain}) = 15$ and $\text{nullity}(A) = 5$: (a) The dimension of the range of T is $\text{rank}(A) = 15 - 5 = 10$. (b) The dimension of the kernel of T is $\text{nullity}(A) = 5$.

4. (10 points) Find the area of the parallelogram(平行四邊形) in \mathbb{R}^3 determined by the vectors $[2, 1, 3]$ and $[4, -3, 1]$. The area is (5) $10\sqrt{3}$.

Solution :

The area of the parallelogram determined by vectors \mathbf{u} and \mathbf{v} is the magnitude of their cross product $\|\mathbf{u} \times \mathbf{v}\|$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 4 & -3 & 1 \end{vmatrix} = \mathbf{i}(1 + 9) - \mathbf{j}(2 - 12) + \mathbf{k}(-6 - 4) = [10, 10, -10]$$

$$\text{Area} = \sqrt{10^2 + 10^2 + (-10)^2} = \sqrt{300} = 10\sqrt{3}$$

5. (10 points) Let P_3 be the vector space of polynomials with degree at most 3 with real coefficients. The coordinate vector of $7x^3 + 3x^2 - 2x + 3$ relative to the ordered basis $(x^2 + x, x^3, x^3 + x, 2x^2 + 1)$ is **(6)** $[-3, 6, 1, 3]^T$.

Solution :

Let the coordinate vector be $[c_1, c_2, c_3, c_4]^T$. We solve:

$$c_1(x^2 + x) + c_2(x^3) + c_3(x^3 + x) + c_4(2x^2 + 1) = 7x^3 + 3x^2 - 2x + 3$$

$$(c_2 + c_3)x^3 + (c_1 + 2c_4)x^2 + (c_1 + c_3)x + c_4 = 7x^3 + 3x^2 - 2x + 3$$

The coordinate vector is $[-3, 6, 1, 3]^T$.

6. (10 points) Suppose that C is a 6×6 matrix with determinant 4. The $\det(7C^{-1})$ is **(7)** $\frac{7^6}{4}$.

Solution :

Using the property $\det(kA) = k^n \det(A)$ for an $n \times n$ matrix, where $n = 6$:

$$\det(7C^{-1}) = 7^6 \cdot \det(C^{-1}) = 7^6 \cdot \frac{1}{\det(C)} = \frac{7^6}{4}$$

7. (10 points)

$$D = \begin{bmatrix} 2 & 4 & -2 & 0 & 5 & -1 & 9 \\ 1 & 2 & -1 & 3 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 & 3 & 1 & 2 \\ 0 & 5 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 4 & 0 & 3 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}, \text{The determinant of } D \text{ is } \textbf{(8) } -60.$$

Solution :

We compute the determinant of D by performing cofactor expansion on columns/rows with the most zeros.

- Expand along the 4th column: $\det(D) = 3 \cdot (-1)^{2+4} \cdot \det(D_{24})$.
- Expand the remaining 6×6 along the 5th row (only element is 2 at position (5,2)):

$$\det(D) = 3 \cdot [(-1)^{5+2} \cdot 2 \cdot \det \begin{pmatrix} 2 & -2 & 5 & -1 & 9 \\ 1 & 1 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}]$$

- Continuing expansion along the 3rd row for 1, then the 4th row for 5:

$$\det(D) = -6 \cdot [1 \cdot 5 \cdot \begin{vmatrix} 2 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix}]$$

- Evaluating the 3×3 determinant: 2.

Final result: $\det(D) = -30 \times 2 = -60$.

8. (10 points) Let P_3 be the vector space of polynomials with degree at most 3 with real coefficients. $T : P_3 \rightarrow P_3$ be defined by $T(p(x)) = 2p(x) - 3\frac{d}{dx}p(x)$

(a) Prove that T is a linear transformation.

(b) Let the ordered basis for P_3 is $B = (1, x + 1, x^2, x^3 - 1)$. Find the matrix representation A of T relative to the ordered bases B .

Answer: (b) $A = \begin{bmatrix} 2 & -3 & 6 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Solution :

Similar with 3-4 example 9.

(a) Proof that T is a linear transformation: To show that T is a linear transformation, we must verify the properties of linear combination. Let $p(x), q(x) \in P_3$ and $r, s \in \mathbb{R}$.

$$\begin{aligned} T(rp(x) + sq(x)) &= 2(rp(x) + sq(x)) - 3\frac{d}{dx}(rp(x) + sq(x)) \\ &= 2rp(x) + 2sq(x) - 3\left(r\frac{d}{dx}p(x) + s\frac{d}{dx}q(x)\right) \\ &= r\left(2p(x) - 3\frac{d}{dx}p(x)\right) + s\left(2q(x) - 3\frac{d}{dx}q(x)\right) \\ &= rT(p(x)) + sT(q(x)) \end{aligned}$$

Since the property holds, T is a linear transformation.

(b) Matrix representation A relative to the basis $B = (1, x + 1, x^2, x^3 - 1)$: To find the matrix $[T]_B$, we apply T to each basis vector and express the result as a linear combination of the basis B .

1. $T(1) = 2(1) - 3(0) = \underline{2} = \underline{\mathbf{2}(1)} + 0(x + 1) + 0(x^2) + 0(x^3 - 1)$
2. $T(x + 1) = 2(x + 1) - 3(1) = \underline{2x - 1} = 2(x + 1) - 3 = \underline{-\mathbf{3}(1)} + \underline{\mathbf{2}(x + 1)} + 0(x^2) + 0(x^3 - 1)$
3. $T(x^2) = 2(x^2) - 3(2x) = \underline{2x^2 - 6x} = \underline{\mathbf{6}(1)} + \underline{-\mathbf{6}(x + 1)} + \underline{\mathbf{2}(x^2)} + 0(x^3 - 1)$
4. $T(x^3 - 1) = 2(x^3 - 1) - 3(3x^2) = \underline{2x^3 - 9x^2 - 2} = \underline{\mathbf{0}(1)} + \underline{\mathbf{0}(x + 1)} + \underline{-\mathbf{9}(x^2)} + \underline{\mathbf{2}(x^3 - 1)}$

Placing the coordinates of these images into the columns of matrix A :

$$A = [T]_B = \begin{bmatrix} 2 & -3 & 6 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

9. (10 points) Let F is the vector space of all functions mapping \mathbb{R} into \mathbb{R} and $S = \{\sin(-x), 1, \sin(x), \sin(2x)\}$.

Is S linear independent in F ? (Yes / No) . If not, find a basis of $\text{sp}(S)$ {1, sin(x), sin(2x)} .

Solution :

注意一下，這邊是計算證明題，需要有解釋，不然沒分。然後第二部分答案不唯一。

Is S linearly independent in F ? No.

The set $S = \{\sin(-x), 1, \sin(x), \sin(2x)\}$. Recall the trigonometric identity for odd functions: $\sin(-x) = -\sin(x)$. This means we can write a non-trivial linear combination that equals the zero function:

$$1 \cdot \sin(-x) + 0 \cdot 1 + 1 \cdot \sin(x) + 0 \cdot \sin(2x) = -\sin(x) + \sin(x) = 0$$

Since there exists a set of coefficients (not all zero) such that the linear combination is zero, the set S is **linearly dependent**.

Find a basis of $\text{span}(S)$:

To find a basis, we remove the redundant vector(s) that can be expressed as a linear combination of others.

1. We observed that $\sin(-x) = -\sin(x)$, so $\sin(-x)$ is in the span of $\{\sin(x)\}$.
2. The remaining set is $\{1, \sin(x), \sin(2x)\}$.
3. We check if these are linearly independent. Consider $c_1(1) + c_2 \sin(x) + c_3 \sin(2x) = 0$ for all $x \in \mathbb{R}$.
 - Let $x = 0$: $c_1(1) + 0 + 0 = 0 \implies c_1 = 0$.
 - Let $x = \pi/2$: $c_2 \sin(\pi/2) + c_3 \sin(\pi) = 0 \implies c_2(1) + 0 = 0 \implies c_2 = 0$.
 - Let $x = \pi/4$: $c_3 \sin(\pi/2) = 0 \implies c_3(1) = 0 \implies c_3 = 0$.

Since $c_1 = c_2 = c_3 = 0$ is the only solution, the set $\{1, \sin(x), \sin(2x)\}$ is linearly independent and spans the same space as S .

Basis of $\text{span}(S)$: $\{1, \sin(x), \sin(2x)\}$

10. (10 points) (a) Build a linear transformation that is one-to-one but not onto.

Solution :

Consider $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by:

$$T \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

- **One-to-one:** The kernel is found by setting $T(\mathbf{x}) = \mathbf{0}$, which implies $x_1 = 0$ and $x_2 = 0$. Since $\ker(T) = \{\mathbf{0}\}$, T is one-to-one.
- **Not onto:** The range of T is the xy -plane in \mathbb{R}^3 . Any vector with a non-zero z -component (e.g., $[0, 0, 1]^T$) has no preimage. Since $\text{rank}(T) = 2 < \dim(\mathbb{R}^3)$, it is not onto.

(b) Build a linear transformation that is onto but not one-to-one.

Solution :

Consider $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by:

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- **Onto:** For any vector $\mathbf{y} = [y_1, y_2]^T \in \mathbb{R}^2$, we can choose $\mathbf{x} = [y_1, y_2, 0]^T$ such that $T(\mathbf{x}) = \mathbf{y}$. Since the range is all of \mathbb{R}^2 , it is onto.
- **Not one-to-one:** The kernel consists of all vectors where $x_1 = 0$ and $x_2 = 0$, specifically $\ker(T) = \{[0, 0, x_3]^T \mid x_3 \in \mathbb{R}\}$. Since the kernel contains non-zero vectors, T is not one-to-one.

* 這兩小題在回答時，記得也要告知 domain 跟 codomain。然後 domain, codomain 本身要是 vector space。

** 另外 mapping 本身也要記得有 linear transformation 的性質（由 2-3 example 3 知道，用矩陣構造不需驗）。

11. (10 points) Determine the set S_1 of all functions f such that $f(0) = 0$ is a subspace in the vector space F of all functions mapping \mathbb{R} into \mathbb{R} .

Answer: Is S_1 a subspace of F ? (Yes / No)

Solution :

1. **Closed under Addition:** Let $f(x), g(x) \in S_1$. By definition, $f(0) = 0$ and $g(0) = 0$. Consider the sum function $(f \oplus g)$:

$$(f \oplus g)(0) = f(0) + g(0) = 0 + 0 = 0 \implies (f \oplus g) \in S_1$$

2. **Closed under Scalar Multiplication:** Let $f(x) \in S_1$ and $r \in \mathbb{R}$. By definition, $f(0) = 0$. Consider the scalar product function $(r \otimes f)$:

$$(r \otimes f)(0) = r \cdot f(0) = r \cdot 0 = 0 \implies (r \otimes f) \in S_1$$

Thus S_1 is a subspace of F .

12. (10 points) Consider the set \mathbb{R}^2 , with the addition defined by $[x, y] \oplus [a, b] = [x + a + 2, y + b]$, and with scalar multiplication defined by $r \otimes [x, y] = [r(x + 2) - 2, ry]$.

a. Is this set a vector space? (Yes / No)

Hint: Show by verifying the closed under two operations, A1-A4 and S1-S4.

b. If the set is a vector space, then find the zero vector and the additive inverse (加法反元素) in this vector space. *Hint:* The zero vector may NOT be the vector $[0, 0]$.

Answer: the zero vector is -2, 0, for any vectors $[x, y]$, the $-[x, y]$ is -x-4, -y

Solution :

這題的零向量跟加法反元素的部分，照定義的做法是這樣：

1. **Find the Zero Vector $\vec{0}$:** Let $\vec{0} = [e_1, e_2]$. We require $[x, y] \oplus [e_1, e_2] = [x, y]$:

$$[x + e_1 + 2, y + e_2] = [x, y]$$

$$\text{Comparing components: } x + e_1 + 2 = x \implies e_1 = -2 \quad y + e_2 = y \implies e_2 = 0$$

Zero vector: $[-2, 0]$

2. **Find the Additive Inverse $-[x, y]$:** Let $-[x, y] = [x', y']$. We require $[x, y] \oplus [x', y'] = \vec{0} = [-2, 0]$:

$$[x + x' + 2, y + y'] = [-2, 0]$$

$$\text{Comparing components: } x + x' + 2 = -2 \implies x' = -x - 4 \quad y + y' = 0 \implies y' = -y$$

Additive inverse: $[-x - 4, -y]$

** 我在題目裡特別給提示了『零向量不是 $[0, 0]$ 』，還是有一堆人踩中這個點！！

或者是用簡單的做法：

By the Theorem 3.1, if (V, \oplus, \otimes) is a vector space, then

1. **Find the Zero Vector $\vec{0}$:**

$$\text{We require } \vec{0} = 0 \otimes [x, y] \implies \vec{0} = 0 \otimes [x, y] = [0(x + 2) - 2, 0y] = [-2, 0].$$

2. **Find the Additive Inverse $-[x, y]$:**

$$\text{We require } -[x, y] = (-1) \otimes [x, y] \implies -[x, y] = (-1) \otimes [x, y] = [-(x + 2) - 2, -y] = [-x - 4, -y].$$

接著驗證十點條件，都對的話，就是 vector space!

Verification of Vector Space Axioms: Let $\vec{u} = [x, y]$, $\vec{v} = [a, b]$, $\vec{w} = [c, d] \in \mathbb{R}^2$ and $r, s \in \mathbb{R}$.

• **Closure under Addition:** $\vec{u} \oplus \vec{v} = [x + a + 2, y + b]$ in \mathbb{R}^2 .

• **Closure under Scalar Multiplication:** $r \otimes \vec{u} = [r(x + 2) - 2, ry]$ in \mathbb{R}^2 .

• **A1 (Associativity of \oplus):**

$$(\vec{u} \oplus \vec{v}) \oplus \vec{w} = [x + a + 2, y + b] \oplus [c, d] = [(x + a + 2) + c + 2, y + b + d] = [x + a + c + 4, y + b + d].$$

$$\vec{u} \oplus (\vec{v} \oplus \vec{w}) = [x, y] \oplus [a + c + 2, b + d] = [x + (a + c + 2) + 2, y + b + d] = [x + a + c + 4, y + b + d]. \quad (\text{Holds})$$

• **A2 (Commutativity of \oplus):** $\vec{u} \oplus \vec{v} = [x + a + 2, y + b] = [a + x + 2, b + y] = \vec{v} \oplus \vec{u}. \quad (\text{Holds})$

- **A3 (Identity Element):** There exists $\vec{0} \oplus \vec{u} = [-2, 0] \oplus [x, y] = [(-2) + x + 2, 0 + y] = [x, y] = \vec{u}$. (Holds)
- **A4 (Inverse Element):** For each \vec{u} , there exists $-\vec{u} = [-x - 4, -y]$ such that $\vec{u} \oplus (-\vec{u}) = \vec{0}$.
 $\vec{u} \oplus (-\vec{u}) = [x, y] \oplus [-x - 4, -y] = [x + (-x - 4) + 2, y + (-y)] = [-2, 0] = \vec{0}$. (Holds)
- **S1 (Scalar Distributivity over Vector Addition):**
 $r \otimes (\vec{u} \oplus \vec{v}) = r \otimes [x + a + 2, y + b] = [r(x + a + 2 + 2) - 2, r(y + b)] = [r(x + a + 4) - 2, r(y + b)]$.
 $(r \otimes \vec{u}) \oplus (r \otimes \vec{v}) = [r(x + 2) - 2, ry] \oplus [r(a + 2) - 2, rb] = [(r(x + 2) - 2) + (r(a + 2) - 2) + 2, ry + rb] = [r(x + 2 + a + 2) - 2, r(y + b)]$. (Holds)
- **S2 (Vector Distributivity over Scalar Addition):**
 $(r + s) \otimes \vec{u} = [(r + s)(x + 2) - 2, (r + s)y] = [r(x + 2) + s(x + 2) - 2, ry + sy]$.
 $(r \otimes \vec{u}) \oplus (s \otimes \vec{u}) = [r(x + 2) - 2, ry] \oplus [s(x + 2) - 2, sy] = [(r(x + 2) - 2) + (s(x + 2) - 2) + 2, ry + sy] = [r(x + 2) + s(x + 2) - 2, ry + sy]$. (Holds)
- **S3 (Associativity of Scalar Multiplication):**
 $r \otimes (s \otimes \vec{u}) = r \otimes [s(x + 2) - 2, sy] = [r(s(x + 2) - 2 + 2) - 2, r(sy)] = [rs(x + 2) - 2, rsy]$.
 $(rs) \otimes \vec{u} = [rs(x + 2) - 2, rsy]$. (Holds)
- **S4 (Scalar Identity):** $1 \otimes \vec{u} = [1(x + 2) - 2, 1(y)] = [x, y]$. (Holds)

