

應數一線性代數 2026 春, 期中考 解答

學號: _____, 姓名: _____

本次考試共有 7 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- 不可翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：誠敬宏遠

誠，一生動念都是誠實端正的。 敬，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 遠，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) **Definitions.**

(a) Given an $n \times n$ matrix A , define the **eigenvalues** and **eigenvectors** of A .

Solution :

Section 5-1, definition 5.2.

(b) State the definition of a **diagonalizable** matrix.

Solution :

Section 5-2, definition 5.3.

這份考卷的第三題其實就有這題的答案了啊！

(c) What is the definition of an **orthogonal matrix**?

Solution :

Section 6-3, definition 6.4.

(d) Let W be a subspace of $\mathbb{R}^n$. Define its **orthogonal complement** $W^\perp$.

Solution :

Section 6-1, definition 6.1.

(e) Let W be a subspace of $\mathbb{R}^n$. Define an **orthonormal basis** for W .

Solution :

Section 6-2, definition 6.3.

2. (50 points) **Fill-in-the-Blanks (10 points each). Only the final answers will be graded.**

(a) Solve the following system of differential equations with initial conditions $x_1(0) = 0, x_2(0) = -3, x_3(0) = 1$:

$$\begin{cases} x_1' = 4x_1 - 2x_3 \\ x_2' = 2x_1 + 5x_2 + 4x_3 \\ x_3' = 5x_3 \end{cases}$$

Answer: $[x_1(t), x_2(t), x_3(t)] = \underline{\text{(I)} [2e^{4t} - 2e^{5t}, -4e^{4t} + e^{5t}, e^{5t}]}$.

Solution :

Rewritten in matrix form as $\vec{x}' = A\vec{x}$, where $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ and eigenvalues of A are 4, 5, 5.

For $\lambda_1 = 4$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. For $\lambda_2 = 5$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

The general solution is:

$$\vec{x}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{4t} - 2c_3 e^{5t} \\ -2c_1 e^{4t} + c_2 e^{5t} \\ c_3 e^{5t} \end{bmatrix}$$

Using the initial conditions at $t = 0$:

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 - 2c_3 \\ -2c_1 + c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Solving this system, we get $c_3 = 1$. Substituting into the first equation gives $c_1 = 2$. Substituting into the second gives $-4 + c_2 = -3 \Rightarrow c_2 = 1$.

Therefore, the specific solution is:

$$\vec{x}(t) = 2e^{4t} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 1e^{5t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1e^{5t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{4t} - 2e^{5t} \\ -4e^{4t} + e^{5t} \\ e^{5t} \end{bmatrix}$$

So, $x_1(t) = 2e^{4t} - 2e^{5t}$, $x_2(t) = -4e^{4t} + e^{5t}$, and $x_3(t) = e^{5t}$.

- (b) Find the least-squares straight line $y = mx + b$ that fits the points $(1, 2), (2, 5), (3, 5)$.

Answer: (II) $y = 1.5x + 1$.

Solution :

Let the line be $y = mx + b$. We can set up the overdetermined system $A\vec{x} = \vec{b}$ as:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}$$

To find the least-squares solution, we solve the normal equations $A^T A \vec{x} = A^T \vec{b}$.

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}, \quad A^T \vec{b} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 2(1) + 5(2) + 5(3) \\ 2(1) + 5(1) + 5(1) \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \end{bmatrix}$$

Now, solve the system $\begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \end{bmatrix}$. Get the answer as $m = \frac{3}{2} = 1.5$, $b = 1$

The least-squares straight line is $y = 1.5x + 1$ (or $y = \frac{3}{2}x + 1$).

(c) Let W be a 4-dimensional subspace of $\mathbb{R}^6$ with an **ordered orthogonal basis**:

$$B = \left(\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -2 \\ 1 \\ 2 \end{bmatrix} \right)$$

Let $\vec{v} \in W$ be a vector with coordinate vector $[2, -3, 6, 1]^T$ relative to B . Find $\|\vec{v}\|$.

Answer: $\|\vec{v}\| = \underline{\text{(III) } 30}$

Solution :

First, we find the squared length of the basis vectors. Since the components of each basis vector are permutations of $\pm 1, \pm 1, \pm 2, \pm 2, \pm 2, \pm 2$, they all have the same length:

$$\|\vec{v}_i\|^2 = 2^2 + 2^2 + 1^2 + 1^2 + (-2)^2 + 2^2 = 18 \quad \text{for } i = 1, 2, 3, 4.$$

The coordinate vector of $\vec{v}$ relative to B is $[2, -3, 6, 1]^T$, which means:

$$\vec{v} = 2\vec{v}_1 - 3\vec{v}_2 + 6\vec{v}_3 + 1\vec{v}_4$$

Since B is an orthogonal basis ($\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$), we can use the generalized Pythagorean theorem to find the squared length of $\vec{v}$:

$$\|\vec{v}\|^2 = 2^2\|\vec{v}_1\|^2 + (-3)^2\|\vec{v}_2\|^2 + 6^2\|\vec{v}_3\|^2 + 1^2\|\vec{v}_4\|^2$$

$$\|\vec{v}\|^2 = (4 + 9 + 36 + 1) \times 18$$

$$\|\vec{v}\|^2 = 50 \times 18 = 900$$

Therefore, the length of $\vec{v}$ is:

$$\|\vec{v}\| = \sqrt{900} = 30$$

- (d) Find the standard matrix A for the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that **projects** vectors onto the line $4x - 5y = 0$.

Answer: $A =$ (IV) $\frac{1}{41} \begin{bmatrix} 25 & 20 \\ 20 & 16 \end{bmatrix}$

Solution :

Method 1

The normal vector of $4x - 5y = 0$ is $\vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. The projection matrix $P = \frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} = \frac{1}{41} \begin{bmatrix} 25 & 20 \\ 20 & 16 \end{bmatrix}$.

Method 2

Let L be the line $4x - 5y = 0$. We can find the standard matrix A of the projection transformation T by determining its eigenvalues and eigenvectors from its geometric behavior:

1. **Vectors on the line:** $\vec{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Thus, $T(\vec{v}_1) = 1 \cdot \vec{v}_1$.
2. **Vectors perpendicular to the line:** $\vec{v}_2 = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$. Thus, $T(\vec{v}_2) = 0 \cdot \vec{v}_2$.

$$C = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & -5 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = CDC^{-1} = \frac{1}{41} \begin{bmatrix} 25 & 20 \\ 20 & 16 \end{bmatrix}$$

- (e) Let A be a 3×3 matrix with $\det(A) = 12$ and $\text{tr}(A) = 8$. If $\lambda_1 = 2$ is an eigenvalue of A , find the other two eigenvalues. Answer: (V) $3 + \sqrt{3}$ and $3 - \sqrt{3}$.

Solution :

Let the other two eigenvalues be λ_2 and λ_3 .

By Chapter 5-1, Problem 37, the trace of a matrix is equal to the sum of its eigenvalues. Therefore, $\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(A) = 8 \implies 2 + \lambda_2 + \lambda_3 = 8 \implies \lambda_2 + \lambda_3 = 6$

By Chapter 5-2, Problem 17, the determinant of a matrix is equal to the product of its eigenvalues. Thus, $\lambda_1 \lambda_2 \lambda_3 = \det(A) = 12 \implies 2\lambda_2 \lambda_3 = 12 \implies \lambda_2 \lambda_3 = 6$

Thus, λ_2 and λ_3 are the roots of the quadratic equation $x^2 - 6x + 6 = 0$.

Using the quadratic formula:

$$x = \frac{6 \pm \sqrt{36 - 24}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$$

The other two eigenvalues are $3 + \sqrt{3}$ and $3 - \sqrt{3}$.

3. (10 points) Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

- (a) Find the eigenvalues of A and their algebraic multiplicities. Answer: 2, 2, 3
- (b) Is A diagonalizable? (Yes / No) . If not, why? for $\lambda = 2$, $\text{GM} = 1 < \text{AM} = 2$.
- (c) Is A orthogonally diagonalizable? (Yes / No) . If not, why? A: not symmetric.
- (d) **If A is diagonalizable**, find A^{100} and matrices D, C such that $A = CDC^{-1}$.

(Note: If A is orthogonally diagonalizable, your matrix C must be an orthogonal matrix; otherwise, provide an invertible matrix C .)

Answer: $C = \underline{\text{X}}$, $D = \underline{\text{X}}$, $A^{100} = \underline{\text{X}}$

Solution :

$$\text{Part (a): } \det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -1 & 1 \\ 1 & 1 - \lambda & -1 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = (3 - \lambda) \begin{vmatrix} 3 - \lambda & -1 \\ 1 & 1 - \lambda \end{vmatrix} = (3 - \lambda)(\lambda - 2)^2.$$

The eigenvalues are $\lambda = 3$ (algebraic multiplicity 1) and $\lambda = 2$ (algebraic multiplicity 2).

Part (b): To check if A is diagonalizable, we must find the geometric multiplicity (GM) of the repeated eigenvalue $\lambda = 2$.

$$A - 2I = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

It is clear that the rank of $(A - 2I)$ is 2. By the Rank-Nullity Theorem, the dimension of the eigenspace (Nullity) is $n - \text{rank} = 3 - 2 = 1$. Since the geometric multiplicity ($\text{GM} = 1$) is strictly less than the algebraic multiplicity ($\text{AM} = 2$), the matrix does not have enough linearly independent eigenvectors. Therefore, A is **not** diagonalizable.

Part (c): By the Spectral Theorem, a real matrix is orthogonally diagonalizable if and only if it is symmetric. It is not symmetric, hence not orthogonally diagonalizable.

Part (d): Since we proved in Part (b) that A is not diagonalizable, we cannot express it as CDC^{-1} . Thus, this part is "Not applicable."

補充一點，這學期的 Quiz 2 的題目考的不一樣喔！那個是問 $AC = CD$ ，這時候的 C 跟 D 是一定存在的 (Theorem 5.2)，但是這題的題幹有說了，如果 A 是 diagonalizable，寫出『invertible』的 C 。

4. (20 points) Consider the matrix A in $\mathbb{R}^4$ below, and let $W = \text{Col}(A)$ be the subspace of $\mathbb{R}^4$ spanned by the columns of A .

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- (a) Find an orthonormal basis for W . Answer: $\frac{1}{\sqrt{2}}[1, 1, 0, 0]^T, \frac{1}{\sqrt{6}}[1, -1, 2, 0]^T, \frac{1}{\sqrt{21}}[-2, 2, 2, 3]^T$

- (b) Find the **QR-factorization** of A and find the **projection matrix** P for W .

$$\text{Answer: } Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -2/\sqrt{21} \\ 1/\sqrt{2} & -1/\sqrt{6} & 2/\sqrt{21} \\ 0 & 2/\sqrt{6} & 2/\sqrt{21} \\ 0 & 0 & 3/\sqrt{21} \end{bmatrix}, R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{21}/3 \end{bmatrix}, P = \frac{1}{7} \begin{bmatrix} 6 & 1 & 1 & -2 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 6 & 2 \\ -2 & 2 & 2 & 3 \end{bmatrix}$$

- (c) Given $\vec{b} = [1, 2, 3, 4]^T$, find the **projection vector** $\vec{b}_W$ of $\vec{b}$ onto W . Answer: $\vec{b}_W = \frac{1}{7} \begin{bmatrix} 3 \\ 18 \\ 25 \\ 20 \end{bmatrix}$

Solution :

Part (a): Gram-Schmidt

$$\vec{v}_1 = \vec{a}_1 = [1, 1, 0, 0]^T \Rightarrow \vec{q}_1 = \frac{1}{\sqrt{2}}[1, 1, 0, 0]^T.$$

$$\vec{v}_2 = \vec{a}_2 - (\vec{q}_1 \cdot \vec{a}_2)\vec{q}_1 = [1, 0, 1, 0]^T - \frac{1}{2}[1, 1, 0, 0]^T = [1/2, -1/2, 1, 0]^T.$$

$$\|\vec{v}_2\| = \sqrt{6}/2 \Rightarrow \vec{q}_2 = \frac{1}{\sqrt{6}}[1, -1, 2, 0]^T.$$

$$\vec{v}_3 = \vec{a}_3 - (\vec{q}_1 \cdot \vec{a}_3)\vec{q}_1 - (\vec{q}_2 \cdot \vec{a}_3)\vec{q}_2 = [0, 1, 1, 1]^T - \frac{1}{2}[1, 1, 0, 0]^T - \frac{1}{6}[1, -1, 2, 0]^T = [-2/3, 2/3, 2/3, 1]^T.$$

$$\|\vec{v}_3\| = \sqrt{21}/3 \Rightarrow \vec{q}_3 = \frac{1}{\sqrt{21}}[-2, 2, 2, 3]^T.$$

Part (b): QR Factorization

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -2/\sqrt{21} \\ 1/\sqrt{2} & -1/\sqrt{6} & 2/\sqrt{21} \\ 0 & 2/\sqrt{6} & 2/\sqrt{21} \\ 0 & 0 & 3/\sqrt{21} \end{bmatrix}, R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{21}/3 \end{bmatrix}$$

Part (c): Projection Matrix

$$P = QQ^T = \frac{1}{7} \begin{bmatrix} 6 & 1 & 1 & -2 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 6 & 2 \\ -2 & 2 & 2 & 3 \end{bmatrix}$$

Part (d): Projection Vector

$$\vec{b}_W = P\vec{b} = \frac{1}{7} \begin{bmatrix} 6 & 1 & 1 & -2 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 6 & 2 \\ -2 & 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 3 \\ 18 \\ 25 \\ 20 \end{bmatrix}$$

5. (10 points) Let A be an $n \times n$ symmetric matrix such that $A^2 = A$.

(a) What are the possible eigenvalues of A ? Answer: 0 or 1

(b) Prove that the column space of A and the null space of A are orthogonal complements (i.e., $\text{Col}(A) = \text{Null}(A)^\perp$).

Solution :

(a) Let λ be an eigenvalue of A with a corresponding non-zero eigenvector $\vec{x}$.

Then $A\vec{x} = \lambda\vec{x}$, which implies $A^2\vec{x} = \lambda^2\vec{x}$.

Since $A^2 = A$, we have $\lambda^2\vec{x} = \lambda\vec{x}$, leading to $(\lambda^2 - \lambda)\vec{x} = \vec{0}$.

Because $\vec{x} \neq \vec{0}$, it must be that $\lambda^2 - \lambda = 0$, which gives $\lambda = 0$ or 1 .

To show that matrices with these eigenvalues actually exist, we can provide the following examples of symmetric matrices satisfying $A^2 = A$:

- For a matrix having both 0 and 1 as eigenvalues, consider $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. The characteristic equation is $\det(B - \lambda I) = (1 - \lambda)(-\lambda) = 0$, thus the eigenvalues of B are 0 and 1.
- Alternatively, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives an eigenvalue of 1, and $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ gives an eigenvalue of 0.

(b) Since A is symmetric, $A = A^T$, which implies that the column space of A is equal to its row space, i.e., $\text{Col}(A) = \text{Row}(A)$.

By the Fundamental Theorem of Linear Algebra, the row space and the null space of a real matrix are orthogonal complements, so $\text{Row}(A) = \text{Null}(A)^\perp$. Therefore, $\text{Col}(A) = \text{Null}(A)^\perp$.

6. (10 points) **True or False.** Circle True or False. If it is false, modify the statement to make it true or provide a counterexample.

- (a) ☒ True ☐ False If A is an orthogonal matrix, then $|\det(A)| = 1$.

Solution :

True. (Since $A^T A = I \Rightarrow \det(A^T) \det(A) = 1 \Rightarrow \det(A)^2 = 1$).

- (b) True ☒ False Every symmetric matrix is orthogonal.

Solution :

False. Correction: Every symmetric matrix is **orthogonally diagonalizable**. (Or provide counterexample: $A = \text{diag}(2, 2)$ is symmetric but not orthogonal).

- (c) True ☒ False If λ is an eigenvalue of a matrix A , then λ is an eigenvalue of the matrix $A + cI$ for all scalars c .

Solution :

False. Correction: $\lambda + c$ is an eigenvalue of $A + cI$.

- (d) True ☒ False Every nonzero vector in $\mathbb{R}^n$ is in some orthonormal basis for $\mathbb{R}^n$.

Solution :

False. Correction: Every **unit** vector in $\mathbb{R}^n$ is in some orthonormal basis. (Or: Every nonzero vector is in some **orthogonal** basis).

- (e) True ☒ False The least-square solution vector of $A\vec{x} = \vec{b}$ is the projection of $\vec{b}$ on the column space of A .

Solution :

False. Correction: $A\vec{x}$ (not $\vec{x}$) is the projection of $\vec{b}$ on the column space of A . (The least-square solution vector $\vec{x}$ is the vector that *produces* the projection).

7. (10 points) Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Prove that there is one and only one vector $\vec{p}$ in W such that $\vec{b} - \vec{p}$ is perpendicular(垂直) to every vector in W .

Solution :

Proof of Uniqueness:

Assume there exist two vectors $\vec{p}_1, \vec{p}_2 \in W$ such that $(\vec{b} - \vec{p}_1) \perp W$ and $(\vec{b} - \vec{p}_2) \perp W$.

This means for all $\vec{w} \in W$, we have $(\vec{b} - \vec{p}_1) \cdot \vec{w} = 0$ and $(\vec{b} - \vec{p}_2) \cdot \vec{w} = 0$.

Subtracting the two equations yields $[(\vec{b} - \vec{p}_2) - (\vec{b} - \vec{p}_1)] \cdot \vec{w} = 0$, which simplifies to $(\vec{p}_1 - \vec{p}_2) \cdot \vec{w} = 0$ for all $\vec{w} \in W$.

This implies that $(\vec{p}_1 - \vec{p}_2) \in W^\perp$ (the orthogonal complement of W).

However, since W is a subspace and $\vec{p}_1, \vec{p}_2 \in W$, their linear combination $(\vec{p}_1 - \vec{p}_2)$ is also in W .

The only vector that belongs to both W and $W^\perp$ is the zero vector, meaning $W \cap W^\perp = \{\vec{0}\}$.

Thus, $\vec{p}_1 - \vec{p}_2 = \vec{0}$, which gives $\vec{p}_1 = \vec{p}_2$. The uniqueness is proved.

學號: _____, 姓名: _____

Answers for Question 2 (Fill-in-the-Blanks):

(I)	(II)	(III)	(VI)	(V)
$\begin{bmatrix} 2e^{4t} - 2e^{5t}, - \\ 4e^{4t} + e^{5t}, e^{5t} \end{bmatrix},$ $\begin{bmatrix} c_1e^{4t} - 2c_3e^{5t} \\ -2c_1e^{4t} + c_2e^{5t} \\ c_3e^{5t} \end{bmatrix},$ $\begin{bmatrix} 2e^{4t} - 2e^{5t} \\ -4e^{4t} + e^{5t} \\ e^{5t} \end{bmatrix}$	$y = 1.5x + 1$	30	$\frac{1}{41} \begin{bmatrix} 25 & 20 \\ 20 & 16 \end{bmatrix}$	$3 \pm \sqrt{3}$

以下由閱卷人員填寫

Question:	1	2	3	4	5	6	7	Total
Points:	10	50	10	20	10	10	10	120
Score:								