應數一線性代數 2026 春, 期中考 解答

學號: ______________________, 姓名: ______________________

本次考試共有 7 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一頁及最後一頁填上姓名學號。
- **不可**翻閱課本或筆記。
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。沒有計算過程，就算回答正確答案也不會得到滿分。答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 書寫空間不夠時，可利用試卷背面，但須標記清楚。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (10 points) **Definitions.**

(a) Given an $n \times n$ matrix A, define the **eigenvalues** and **eigenvectors** of A.

Solution：

A scalar λ is called an **eigenvalue** of an $n \times n$ matrix A if there exists a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$. The vector $\vec{v}$ is called an **eigenvector** of A corresponding to λ.

(b) State the definition of a **diagonalizable** matrix.

Solution：

An $n \times n$ matrix A is said to be **diagonalizable** if there exists an invertible matrix C and a diagonal matrix D such that $D = C^{-1}AC$ (or equivalently, $A = CDC^{-1}$).

(c) What is the definition of an **orthogonal matrix**?

Solution：

A square matrix A is an **orthogonal matrix** if its transpose is equal to its inverse, i.e., $A^T = A^{-1}$. This is equivalent to saying that the columns of A form an orthonormal basis for $\mathbb{R}^n$.

(d) Let W be a subspace of $\mathbb{R}^n$. Define its **orthogonal complement** $W^\perp$.

Solution：

The **orthogonal complement** $W^\perp$ is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W. (Or in set notation: $W^\perp = \{\vec{v} \in \mathbb{R}^n \mid \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$)

(e) Let W be a subspace of $\mathbb{R}^n$. Define an **orthonormal basis** for W.

Solution：

An **orthonormal basis** for a subspace W is an orthogonal basis where every vector in the set has a unit length (norm of 1).

2. (50 points) **Fill-in-the-Blanks (10 points each). Only the final answers will be graded.**

(a) Solve the following system of differential equations with initial conditions $x_1(0) = 0, x_2(0) = -3, x_3(0) = 1$:

$$\begin{cases} x_1' = 4x_1 - 2x_3 \\ x_2' = 2x_1 + 5x_2 + 4x_3 \\ x_3' = 5x_3 \end{cases}$$

Answer: $[x_1(t), x_2(t), x_3(t)] =$ **(I)** $[2e^{4t} - 2e^{5t},\ -4e^{4t} + e^{5t},\ e^{5t}]$.

Solution：

Rewritten in matrix form as $\vec{x}' = A\vec{x}$, where $A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ and eigenvalues of A are 4, 5, 5.

For $\lambda_1 = 4$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$. For $\lambda_2 = 5$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

The general solution is:

$$\vec{x}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{5t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Using the initial conditions at $t = 0$:

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 - 2c_3 \\ -2c_1 + c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

Solving this system, we get $c_3 = 1$. Substituting into the first equation gives $c_1 = 2$. Substituting into the second gives $-4 + c_2 = -3 \Rightarrow c_2 = 1$.

Therefore, the specific solution is:

$$\vec{x}(t) = 2e^{4t} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 1e^{5t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1e^{5t} \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e^{4t} - 2e^{5t} \\ -4e^{4t} + e^{5t} \\ e^{5t} \end{bmatrix}$$

So, $x_1(t) = 2e^{4t} - 2e^{5t}$, $x_2(t) = -4e^{4t} + e^{5t}$, and $x_3(t) = e^{5t}$.

(b) Find the least-squares straight line $y = mx + b$ that fits the points $(1, 2), (2, 5), (3, 5)$.

Answer: **(II)** $y = 1.5x + 1$.

Solution：

Let the line be $y = mx + b$. We can set up the overdetermined system $A\vec{x} = \vec{b}$ as:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix}$$

To find the least-squares solution, we solve the normal equations $A^T A\vec{x} = A^T\vec{b}$.

$$A^T A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix}, \ A^T\vec{b} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 2(1) + 5(2) + 5(3) \\ 2(1) + 5(1) + 5(1) \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \end{bmatrix}$$

Now, solve the system $\begin{bmatrix} 14 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 27 \\ 12 \end{bmatrix}$. Get the answer as $m = \frac{3}{2} = 1.5, \ b = 1$

The least-squares straight line is $y = 1.5x + 1$ (or $y = \frac{3}{2}x + 1$).

(c) Let W be a 4-dimensional subspace of $\mathbb{R}^6$ with an **ordered orthogonal basis**:

$$B = \left(\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \\ -2 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -2 \\ 1 \\ 2 \end{bmatrix} \right)$$

Let $\vec{v} \in W$ be a vector with coordinate vector $[2, -3, 6, 1]^T$ relative to B. Find $\|\vec{v}\|$.

Answer: $\|\vec{v}\| =$ **(III) 30**

Solution：

First, we find the squared length of the basis vectors. Since the components of each basis vector are permutations of $\pm1, \pm1, \pm2, \pm2, \pm2, \pm2$, they all have the same length:

$$\|\vec{v}_i\|^2 = 2^2 + 2^2 + 1^2 + 1^2 + (-2)^2 + 2^2 = 18 \quad \text{for } i = 1, 2, 3, 4.$$

The coordinate vector of $\vec{v}$ relative to B is $[2, -3, 6, 1]^T$, which means:

$$\vec{v} = 2\vec{v}_1 - 3\vec{v}_2 + 6\vec{v}_3 + 1\vec{v}_4$$

Since B is an orthogonal basis ($\vec{v}_i \cdot \vec{v}_j = 0$ for $i \neq j$), we can use the generalized Pythagorean theorem to find the squared length of $\vec{v}$:

$$\|\vec{v}\|^2 = 2^2\|\vec{v}_1\|^2 + (-3)^2\|\vec{v}_2\|^2 + 6^2\|\vec{v}_3\|^2 + 1^2\|\vec{v}_4\|^2$$

$$\|\vec{v}\|^2 = (4 + 9 + 36 + 1) \times 18$$

$$\|\vec{v}\|^2 = 50 \times 18 = 900$$

Therefore, the length of $\vec{v}$ is:

$$\|\vec{v}\| = \sqrt{900} = 30$$

(d) Find the standard matrix A for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that projects vectors onto the line $4x - 5y = 0$.

Answer: $A =$ **(IV)** $\frac{1}{41}\begin{bmatrix} 25 & 20 \\ 20 & 16 \end{bmatrix}$

Solution：

Method 1

The normal vector of $4x - 5y = 0$ is $\vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. The projection matrix $P = \frac{\vec{v}\vec{v}^T}{\vec{v}^T\vec{v}} = \frac{1}{41}\begin{bmatrix} 25 & 20 \\ 20 & 16 \end{bmatrix}$.

Method 2

Let L be the line $4x - 5y = 0$. We can find the standard matrix A of the projection transformation T by determining its eigenvalues and eigenvectors from its geometric behavior:

1. **Vectors on the line:** $\vec{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Thus, $T(\vec{v}_1) = 1 \cdot \vec{v}_1$.
2. **Vectors perpendicular to the line:** $\vec{v}_2 = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$. Thus, $T(\vec{v}_2) = 0 \cdot \vec{v}_2$.

$$C = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & -5 \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; P = CDC^{-1} = \frac{1}{41}\begin{bmatrix} 25 & 20 \\ 20 & 16 \end{bmatrix}$$

(e) Let A be a 3×3 matrix with $\det(A) = 12$ and $tr(A) = 8$. If $\lambda_1 = 2$ is an eigenvalue of A, find the other two eigenvalues. Answer: **(V)** $\mathbf{3 + \sqrt{3}}$ **and** $\mathbf{3 - \sqrt{3}}$.

Solution：

Let the other two eigenvalues be λ_2 and λ_3.

By Chapter 5-1, Problem 37, the trace of a matrix is equal to the sum of its eigenvalues. Therefore, $\lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(A) = 8 \implies 2 + \lambda_2 + \lambda_3 = 8 \implies \lambda_2 + \lambda_3 = 6$

By Chapter 5-2, Problem 17, the determinant of a matrix is equal to the product of its eigenvalues. Thus, $\lambda_1\lambda_2\lambda_3 = \det(A) = 12 \implies 2\lambda_2\lambda_3 = 12 \implies \lambda_2\lambda_3 = 6$

Thus, λ_2 and λ_3 are the roots of the quadratic equation $x^2 - 6x + 6 = 0$.

Using the quadratic formula:

$$x = \frac{6 \pm \sqrt{36 - 24}}{2} = \frac{6 \pm 2\sqrt{3}}{2} = 3 \pm \sqrt{3}$$

The other two eigenvalues are $\mathbf{3 + \sqrt{3}}$ and $\mathbf{3 - \sqrt{3}}$.

3. (10 points) Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Find the eigenvalues of A and their algebraic multiplicities. Answer: **2, 2, 3**

(b) Is A diagonalizable? (Yes / $\boxed{No}$) . If not, why? **for $\lambda = 2$, GM = 1 < AM = 2.**

(c) Is A orthogonally diagonalizable? (Yes / $\boxed{No}$) . If not, why? **A: not symmetric.**

(d) **If** A is diagonalizable, find A^{100} and matrices D, C such that $A = CDC^{-1}$.

(**Note:** If A is orthogonally diagonalizable, your matrix C must be an orthogonal matrix; otherwise, provide an invertible matrix C.)

Answer: $C = \begin{bmatrix} 1 & 0 & -2 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, $A^{100} = \begin{bmatrix} 4^{100} & 0 & 2(4^{100} - 5^{100}) \\ 2(5^{100} - 4^{100}) & 5^{100} & 4(5^{100} - 4^{100}) \\ 0 & 0 & 5^{100} \end{bmatrix}$

Solution：

Part (a): $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (3-\lambda)(\lambda-2)^2.$

The eigenvalues are $\lambda = 3$ (algebraic multiplicity 1) and $\lambda = 2$ (algebraic multiplicity 2).

Part (b): To check if A is diagonalizable, we must find the geometric multiplicity (GM) of the repeated eigenvalue $\lambda = 2$.

$$A - 2I = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

It is clear that the rank of $(A - 2I)$ is 2. By the Rank-Nullity Theorem, the dimension of the eigenspace (Nullity) is $n - \text{rank} = 3 - 2 = 1$. Since the geometric multiplicity (GM = 1) is strictly less than the algebraic multiplicity (AM = 2), the matrix does not have enough linearly independent eigenvectors. Therefore, A is **not** diagonalizable.

Part (c): By the Spectral Theorem, a real matrix is orthogonally diagonalizable if and only if it is symmetric. It is not symmetric, hence not orthogonally diagonalizable.

Part (d): Since we proved in Part (b) that A is not diagonalizable, we cannot express it as CDC^{-1}. Thus, this part is "Not applicable."

4. (20 points) Consider the matrix A in $\mathbb{R}^4$ below, and et $W = \text{Col}(A)$ be the subspace of $\mathbb{R}^4$ spanned by the columns of A.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Find an orthonormal basis for W. Answer: $\frac{1}{\sqrt{2}}[1,1,0,0]^T, \frac{1}{\sqrt{6}}[1,-1,2,0]^T, \frac{1}{\sqrt{21}}[-2,2,2,3]^T$

(b) Find the **QR-factorization** of A and find the **projection matrix** P for W.
Answer: $Q =$ ____ , $R =$ ____ , $P =$ ____ .

(c) Given $\vec{b} = [1,2,3,4]^T$, find the **projection vector** $\vec{b}_W$ of $\vec{b}$ onto W. Answer: $\vec{b}_W =$ ____

Solution：

Part (a): Gram-Schmidt

$\vec{v}_1 = \vec{a}_1 = [1,1,0,0]^T \Rightarrow \vec{q}_1 = \frac{1}{\sqrt{2}}[1,1,0,0]^T$.
$\vec{v}_2 = \vec{a}_2 - (\vec{q}_1 \cdot \vec{a}_2)\vec{q}_1 = [1,0,1,0]^T - \frac{1}{2}[1,1,0,0]^T = [1/2,-1/2,1,0]^T$.
$\|\vec{v}_2\| = \sqrt{6}/2 \Rightarrow \vec{q}_2 = \frac{1}{\sqrt{6}}[1,-1,2,0]^T$.
$\vec{v}_3 = \vec{a}_3 - (\vec{q}_1 \cdot \vec{a}_3)\vec{q}_1 - (\vec{q}_2 \cdot \vec{a}_3)\vec{q}_2 = [0,1,1,1]^T - \frac{1}{2}[1,1,0,0]^T - \frac{1}{6}[1,-1,2,0]^T = [-2/3,2/3,2/3,1]^T$.
$\|\vec{v}_3\| = \sqrt{21}/3 \Rightarrow \vec{q}_3 = \frac{1}{\sqrt{21}}[-2,2,2,3]^T$.

Part (b): QR Factorization

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -2/\sqrt{21} \\ 1/\sqrt{2} & -1/\sqrt{6} & 2/\sqrt{21} \\ 0 & 2/\sqrt{6} & 2/\sqrt{21} \\ 0 & 0 & 3/\sqrt{21} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3/2} & 1/\sqrt{6} \\ 0 & 0 & \sqrt{21}/3 \end{bmatrix}$$

Part (c): Projection Matrix

$$P = QQ^T = \frac{1}{7}\begin{bmatrix} 6 & 1 & 1 & -2 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 6 & 2 \\ -2 & 2 & 2 & 3 \end{bmatrix}$$

Part (d): Projection Vector

$$\vec{b}_W = P\vec{b} = \frac{1}{7}\begin{bmatrix} 6 & 1 & 1 & -2 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 6 & 2 \\ -2 & 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 \\ 18 \\ 25 \\ 20 \end{bmatrix}$$

5. (10 points) Let A be a $n \times n$ symmetric matrix such that $A^2 = A$.

   (a) What are the possible eigenvalues of A?

   (b) Prove that the column space of A and the null space of A are orthogonal complements (i.e., $\text{Col}(A) = \text{Null}(A)^\perp$).

   Solution：

   (a) $A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = \lambda^2\vec{x}$. Since $A^2 = A$, $\lambda^2\vec{x} = \lambda\vec{x} \Rightarrow (\lambda^2 - \lambda)\vec{x} = \vec{0}$. Since $\vec{x} \neq \vec{0}$, $\lambda = 0$ or 1.
   (b) Because A is symmetric, $A = A^T$, which means $\text{Col}(A) = \text{Row}(A)$. By the Fundamental Theorem of Linear Algebra, the row space and null space are orthogonal complements, so $\text{Row}(A) = \text{Null}(A)^\perp$. Therefore, $\text{Col}(A) = \text{Null}(A)^\perp$.

6. (10 points) **True or False.** Circle True or False. **If it is false, modify the statement to make it true or provide a counterexample.**

   (a) **True** False If A is an orthogonal matrix, then $|\det(A)| = 1$.

   Solution：

   True. (Since $A^T A = I \Rightarrow \det(A^T)\det(A) = 1 \Rightarrow \det(A)^2 = 1$).

   (b) True **False** Every symmetric matrix is orthogonal.

   Solution：

   False. Correction: Every symmetric matrix is **orthogonally diagonalizable**. (Or provide counterexample: $A = \text{diag}(2, 2)$ is symmetric but not orthogonal).

   (c) True **False** If λ is an eigenvalue of a matrix A, then λ is an eigenvalue of the matrix $A + cI$ for all scalars c.

   Solution：

   False. Correction: $\lambda + c$ is an eigenvalue of $A + cI$.

   (d) True **False** Every nonzero vector in $\mathbb{R}^n$ is in some orthonormal basis for $\mathbb{R}^n$.

   Solution：

   False. Correction: Every **unit** vector in $\mathbb{R}^n$ is in some orthonormal basis. (Or: Every nonzero vector is in some **orthogonal** basis).

   (e) True **False** The least-square solution vector of $A\vec{x} = \vec{b}$ is the projection of $\vec{b}$ on the column space of A.

   Solution：

   False. Correction: $A\vec{x}$ (not $\vec{x}$) is the projection of $\vec{b}$ on the column space of A. (The least-square solution vector $\vec{x}$ is the vector that *produces* the projection).

7. (10 points) Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Prove that there is one and only one vector $\vec{p}$ in W such that $\vec{b}-\vec{p}$ is perpendicular(垂直) to every vector in W.

Solution：

Proof of Uniqueness:

Assume there exist two vectors $\vec{p}_1, \vec{p}_2 \in W$ such that $(\vec{b}-\vec{p}_1) \perp W$ and $(\vec{b}-\vec{p}_2) \perp W$.
This means for all $\vec{w} \in W$, we have $(\vec{b}-\vec{p}_1) \cdot \vec{w} = 0$ and $(\vec{b}-\vec{p}_2) \cdot \vec{w} = 0$.
Subtracting the two equations yields $[(\vec{b}-\vec{p}_2)-(\vec{b}-\vec{p}_1)] \cdot \vec{w} = 0$, which simplifies to $(\vec{p}_1-\vec{p}_2) \cdot \vec{w} = 0$ for all $\vec{w} \in W$.
This implies that $(\vec{p}_1-\vec{p}_2) \in W^{\perp}$ (the orthogonal complement of W).

However, since W is a subspace and $\vec{p}_1, \vec{p}_2 \in W$, their linear combination $(\vec{p}_1-\vec{p}_2)$ is also in W.
The only vector that belongs to both W and $W^{\perp}$ is the zero vector, meaning $W \cap W^{\perp} = \{\vec{0}\}$.
Thus, $\vec{p}_1-\vec{p}_2 = \vec{0}$, which gives $\vec{p}_1 = \vec{p}_2$. The uniqueness is proved.

學號: ______________________, 姓名: ______________________

Answers for Question 2 (Fill-in-the-Blanks):

(I)	**(II)**	**(III)**	**(VI)**	**(V)**
$[2e^{4t}-2e^{5t},\ -4e^{4t}+e^{5t},\ e^{5t}]$	$y = 1.5x+1$	30	$\frac{1}{41}\begin{bmatrix}25 & 20\\ 20 & 16\end{bmatrix}$	$\mathbf{3 \pm \sqrt{3}}$

以下由閱卷人員填寫

Question:	1	2	3	4	5	6	7	Total
Points:	10	50	10	20	10	10	10	120
Score:								