應數一線性代數 2026 春, 期末考

學號: ________________________________, 姓名: ________________________________

本次考試共有 8 頁 (包含封面)，有 8 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本、筆記或使用手機。
- 非考試必須物品請放置於教室前後（手機關靜音），**違者作弊論！**
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 填充題請在紙上留下必要的計算過程，雖然過程不算分，但如果答案正確，卷子卻是白的，答案可能會不算分。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (20 points) Construction based on Definitions

For each of the following conditions, provide a specific example that satisfies the requirement. **If no such example exists, write "Impossible" and briefly explain why based on definitions.**

(a) Let $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in $\mathbb{R}^2$. Find an ordered basis $B = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ such that its coordinate vector is $[\vec{v}]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Provide two matrices A and B, where they have the exact same eigenvalues, but A is **NOT similar** to B.

(c) A matrix that is **Unitarily diagonalizable** but NOT **Hermitian**.

(d) A matrix that is **Unitary** but NOT **Diagonalizable**.

(e) A matrix in **Jordan Canonical Form** where the Algebraic Multiplicity (A.M.) of its only eigenvalue is 4, but its Geometric Multiplicity (G.M.) is 3.

2. (50 points) **Fill-in-the-Blanks (10 points each). Only the final answers will be graded.**

(a) Calculate the exact value of the complex number in standard form $a+bi$:
$(\sqrt{3}-i)^6 =$ **(I)** .

(b) Let V be a vector space with ordered bases $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $B' = \{\vec{b}'_1, \vec{b}'_2, \vec{b}'_3\}$. If

$$C_{B,B'} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \\ 1 & 0 & 4 \end{bmatrix}, \text{ and } \vec{v} = -\vec{b}_1 + 2\vec{b}_2 + 2\vec{b}_3$$

Find the coordinate vector $[\vec{v}]_{B'} =$ **(II)**

(c) Consider $u = \begin{bmatrix} 2-i \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -i \\ 1+i \end{bmatrix}$. Compute the standard Hermitian inner product $\langle u, v \rangle =$ **(III)** .

(d) Let P_2 be the space of polynomials of degree at most 2. Let $B = \{1, x, x^2\}$ be the standard basis. Define a linear transformation $T : P_2 \to P_2$ by $T(p(x)) = 2xp'(x) - p(x)$. The matrix representation $[T]_B$ is **(IV)** .

(e) Let A be a 3×3 complex matrix with $\det(A) = 5 - 2i$, then $\det(2iA) =$ **(V)** .

3. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation and $B = ([1,2],\ [2,-1])$ and $B' = ([1,0,1],\ [0,1,1],\ [1,1,0])$ be ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Suppose that the matrix representation $R_{B,B'}$ of T is given by

$$R_{B,B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 2 \end{bmatrix}$$

Please express $T([4,3])$ and $T([-3,4])$ as vectors in $\mathbb{R}^3$.

Answer: $T([4,3]) =$ ____________________, and $T([-3,4]) =$ ____________________.

4. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} 10 & 0 & 2 & 0 & 0 & 0 \\ 6 & 7 & 3 & 6 & -1 & 0 \\ -8 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 6 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

(a) Jordan canonical form = ____________________, Jordan basis = ____________________

(b) Find the $\det(A^{50}) =$ ____.

Notice that

$$(A-6I) \sim \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (A-6I)^2 = \begin{bmatrix} 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (A-6I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. (10 points) Let A be a 5×5 matrix over $\mathbb{C}$. You are given the following information:

- The characteristic polynomial of A is $p(\lambda) = (\lambda - 2)^5$.
- The eigenspace corresponding to $\lambda = 2$, denoted as E_2, has dimension 2. (i.e., $\dim(Null(A - 2I)) = 2$).
- The rank of the matrix $(A - 2I)^2$ is 1.

Find the Jordan Canonical Form J of the matrix A. Explain your reasoning clearly. *(No extensive matrix computation is needed).*

6. (10 points) Using the Gram-Schmidt process to transform the basis $\{[1,\ i,\ 0], [1,\ 2i,\ 1]\}$ into an orthogonal basis and then extend it as an orthogonal basis for $\mathbb{C}^3$.

   Answer: the found orthogonal basis for $\mathbb{C}^3$ is ________________

7. (10 points) Let $U \in \mathbb{C}^{n\times n}$ be a **Unitary** matrix. Let $V = \frac{3+4i}{5}U$.
   Prove that V is also a Unitary matrix.

8. (10 points) Prove that a normal matrix is Hermitian if and only if all its eigenvalues are in $\mathbb{R}$.

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Answers for Question 2 (Fill-in-the-Blanks):

(I)	**(II)**	**(III)**	**(IV)**	**(V)**

以下由閱卷人員填寫

Question:	1	2	3	4	5	6	7	8	Total
Points:	20	50	10	10	10	10	10	10	130
Score:									