應數一線性代數 2026 春, 期末考 解答

學號: ________________________, 姓名: ________________________

本次考試共有 8 頁 (包含封面)，有 8 題。如有缺頁或漏題，請立刻告知監考人員。

考試須知:

- 請在第一及最後一頁填上姓名學號，並在每一頁的最上方屬名，避免釘書針斷裂後考卷遺失。
- **不可**翻閱課本、筆記或使用手機。
- 非考試必須物品請放置於教室前後（手機關靜音），**違者作弊論！**
- 計算題請寫出計算過程，閱卷人員會視情況給予部份分數。
  沒有計算過程，就算回答正確答案也不會得到滿分。
  答卷請清楚乾淨，儘可能標記或是框出最終答案。
- 填充題請在紙上留下必要的計算過程，雖然過程不算分，但如果答案正確，卷子卻是白的，答案可能會不算分。

高師大校訓：**誠敬宏遠**

誠，一生動念都是誠實端正的。 **敬**，就是對知識的認真尊重。
宏，開拓視界，恢宏心胸。 **遠**，任重致遠，不畏艱難。

請尊重自己也尊重其他同學，考試時請勿東張西望交頭接耳。

1. (20 points) Construction based on Definitions
For each of the following conditions, provide a specific example that satisfies the requirement. **If no such example exists, write "Impossible" and briefly explain why based on definitions.**

(a) Let $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ in $\mathbb{R}^2$. Find an ordered basis $B = \{\vec{b}_1, \vec{b}_2\}$ for $\mathbb{R}^2$ such that its coordinate vector is $[\vec{v}]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Provide two matrices A and B, where they have the exact same eigenvalues, but A is **NOT similar** to B.

(c) A matrix that is **Unitarily diagonalizable** but NOT **Hermitian**.

(d) A matrix that is **Unitary** but NOT **Diagonalizable**.

(e) A matrix in **Jordan Canonical Form** where the Algebraic Multiplicity (A.M.) of its only eigenvalue is 4, but its Geometric Multiplicity (G.M.) is 3.

Solution：

這整題都算是定義題，所以我不想給答案，如果自己找不到，再跟我詢問。

2. (50 points) **Fill-in-the-Blanks (10 points each). Only the final answers will be graded.**

(a) Calculate the exact value of the complex number in standard form $a + bi$:
$(\sqrt{3} - i)^6 =$ __**(I) -64**__ .

Solution：
使用 De Moivre's Theorem：$\sqrt{3} - i = 2(\cos\frac{-\pi}{6} + i\sin\frac{-\pi}{6}) = 2e^{-i\pi/6}$。六次方為 $2^6 \cdot e^{-i\pi} = 64 \times (-1) = -64$。

(b) Let V be a vector space with ordered bases $B = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ and $B' = \{\vec{b}'_1, \vec{b}'_2, \vec{b}'_3\}$. If

$$C_{B,B'} = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & -2 \\ 1 & 0 & 4 \end{bmatrix}, \text{ and } \vec{v} = -\vec{b}_1 + 2\vec{b}_2 + 2\vec{b}_3$$

Find the coordinate vector $[\vec{v}]_{B'} =$ **(II)** $\begin{bmatrix} -2 \\ 2 \\ 7 \end{bmatrix}$

Solution：
$[\vec{v}]_B = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$。計算 $[\vec{v}]_{B'} = C_{B,B'}[\vec{v}]_B = \begin{bmatrix} -2-2+2 \\ 0+6-4 \\ -1+0+8 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 7 \end{bmatrix}$。

(c) Consider $u = \begin{bmatrix} 2-i \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} -i \\ 1+i \end{bmatrix}$. Compute the standard Hermitian inner product $\langle u, v \rangle =$ __**(III)** $4+i$__ .

Solution：
$\langle u, v \rangle = \overline{(2-i)}(-i) + \overline{3}(1+i) = (2+i)(-i) + 3(1+i) = -2i - i^2 + 3 + 3i = 1 - 2i + 3 + 3i = 4 + i.$

(d) Let P_2 be the space of polynomials of degree at most 2. Let $B = \{1, x, x^2\}$ be the standard basis. Define a linear transformation $T : P_2 \to P_2$ by $T(p(x)) = 2xp'(x) - p(x)$. The matrix representation $[T]_B$ is __**(IV) 3**__ .

Solution：
$T(1) = -1$, $T(x) = 2x(1) - x = x$, $T(x^2) = 2x(2x) - x^2 = 3x^2$。矩陣為對角矩陣 diag(-1, 1, 3)，Trace = -1 + 1 + 3 = 3。

(e) Let A be a 3×3 complex matrix with $\det(A) = 5 - 2i$, then $\det(2iA) =$ __**(V)** $-16 - 40i$__ .

Solution：
因為 A 是 3×3，$\det(2iA) = (2i)^3 \det(A) = -8i(5 - 2i) = -40i + 16i^2 = -16 - 40i$。

3. (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation and $B = ([1,2],\ [2,-1])$ and $B' = ([1,0,1],\ [0,1,1],\ [1,1,0])$ be ordered bases of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Suppose that the matrix representation $R_{B,B'}$ of T is given by

$$R_{B,B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 2 \end{bmatrix}$$

Please express $T([4,3])$ and $T([-3,4])$ as vectors in $\mathbb{R}^3$.

Answer: $T([4,3]) =$ ___$[7,3,6]$___ , and $T([-3,4]) =$ ___$[-4,-11,-7]$___ .

Solution：

By definition, $(T(\vec{v}))_{B'} = R_{B,B'}\vec{v}_B$.

(a) For $[4,3]$: $[4,3] = 2[1,2] + 1[2,-1] \implies [4,3]_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$(T([4,3]))_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}.$$

$T([4,3]) = 5[1,0,1] + 1[0,1,1] + 2[1,1,0] = [7,3,6]$.

(b) For $[-3,4]$: $[-3,4] = 1[1,2] - 2[2,-1] \implies [-3,4]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

$$(T([-3,4]))_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -7 \\ -4 \end{bmatrix}.$$

$T([-3,4]) = 0[1,0,1] - 7[0,1,1] - 4[1,1,0] = [-4,-11,-7]$.

4. (10 points) Find a Jordan canonical form and a Jordan basis for the matrix A

$$A = \begin{bmatrix} 10 & 0 & 2 & 0 & 0 & 0 \\ 6 & 7 & 3 & 6 & -1 & 0 \\ -8 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 6 & 1 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

(a) Jordan canonical form = $\underline{\ J\ }$, Jordan basis = **ordered columns of C (不唯一)**

(b) Find the $\det(A^{50}) = \underline{(6^6)^{50} = 6^{300}}$.

Notice that

$$(A-6I) \sim \begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (A-6I)^2 = \begin{bmatrix} 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, (A-6I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution：

Similar with 113-2 exam 2 problem 7.

$$\begin{aligned} (A-6I): \quad & \vec{b}_2 \to \vec{b}_1 \to \vec{0} \\ & \vec{b}_5 \to \vec{b}_4 \to \vec{b}_3 \to \vec{0} \\ & \vec{b}_6 \to \vec{0} \end{aligned}$$

$$C = \begin{bmatrix} 0 & 0 & -4 & 0 & 0 & 0 \\ -1 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 8 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 6 & 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}, \quad A = CJC^{-1}.$$

(b) $\det(A) = 6^6$, so $\det(A^{50}) = (\det A)^{50} = (6^6)^{50} = 6^{300}$.

5. (10 points) Let A be a 5×5 matrix over $\mathbb{C}$. You are given the following information:

- The characteristic polynomial of A is $p(\lambda) = (\lambda - 2)^5$.
- The eigenspace corresponding to $\lambda = 2$, denoted as E_2, has dimension 2. (i.e., $\dim(Null(A - 2I)) = 2$).
- The rank of the matrix $(A - 2I)^2$ is 1.

Find the Jordan Canonical Form J of the matrix A. Explain your reasoning clearly. *(No extensive matrix computation is needed).*

Solution：

1. The characteristic polynomial of A is $p(\lambda) = (\lambda - 2)^5$, imply A has only one eigenvalue $\lambda = 2$ and $A.M. = 5$. The matrix J has only number 2 in its diagonal.
2. $\dim(E_2) = 2$, imply nullity of $(A - 2I) = 2$.
3. The rank of the matrix $(A - 2I)^2$ is 1. Since [rank + nullity = 5], the nullity of $(A - 2I)^2 = 4$.
4. Without state, we can easily get the nullity of $(A - 2I)^3 = 5$.

Thus the Jordan basis should form as: (特別注意，這邊不是 standard basis $\vec{e}$，而是 Jordan basis $\vec{b}$)

$$\vec{b}_3 \to \vec{b}_2 \to \vec{b}_1 \to \vec{0}$$
$$\vec{b}_5 \to \vec{b}_4 \to \vec{0}$$

Jordan Canonical Form is $J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$ (區塊的順序可以調換)

6. (10 points) Using the Gram-Schmidt process to transform the basis $\{[1,\ i,\ 0], [1,\ 2i,\ 1]\}$ into an orthogonal basis and then extend it as an orthogonal basis for $\mathbb{C}^3$.

   Answer: the found orthogonal basis for $\mathbb{C}^3$ is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ **(不唯一)**

   Solution：

   Let $\vec{a}_1 = [1, i, 0]$, $\vec{a}_2 = [1, 2i, 1]$.

   Let $\vec{v}_1 = \vec{a}_1 = [1, i, 0]$. $\|\vec{v}_1\|^2 = 1^2 + |i|^2 = 2$.

   Let $\vec{v}_2 = \vec{a}_2 - \frac{\langle \vec{v}_1, \vec{a}_2 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle}\vec{v}_1 = \vec{a}_2 - \frac{3}{2}\vec{v}_1 = [1, 2i, 1] - [\frac{3}{2}, \frac{3}{2}i, 0] = [-\frac{1}{2}, \frac{1}{2}i, 1]$.

   Let $\vec{v}_3 = \begin{vmatrix} i & j & k \\ 1 & -i & 0 \\ -1 & -i & 2 \end{vmatrix} = [-2i, -2, -2i]$。

7. (10 points) Let $U \in \mathbb{C}^{n \times n}$ be a **Unitary** matrix. Let $V = \frac{3+4i}{5}U$.
   Prove that V is also a Unitary matrix.

   Solution：

   證明： By definition, a matrix is unitary if $V^*V = I$. We compute V^*V:

$$\begin{aligned} V^*V &= (\frac{3+4i}{5}U)^*(\frac{3+4i}{5}U) \\ &= (\overline{\frac{3+4i}{5}}U^*)(\frac{3+4i}{5}U) \\ &= (\frac{3-4i}{5}U^*)(\frac{3+4i}{5}U) \\ &= (\frac{3-4i}{5})(\frac{3+4i}{5})U^*U \\ &= (1)I \quad (\text{since } U \text{ is unitary, } U^*U = I) \\ &= I \end{aligned}$$

   Since $V^*V = I$, V is a unitary matrix.

8. (10 points) Prove that a normal matrix is Hermitian if and only if all its eigenvalues are in $\mathbb{R}$.

Solution：

From 9-3 #24，先用 Theorem 9.7，再往下證會簡單很多。

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Answers for Question 2 (Fill-in-the-Blanks):

(I)	(II)	(III)	(IV)	(V)
-64	$\begin{bmatrix} -2 \\ 2 \\ 7 \end{bmatrix}$	$4+i$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$	$-16-40i$

以下由閱卷人員填寫

Question:	1	2	3	4	5	6	7	8	Total
Points:	20	50	10	10	10	10	10	10	130
Score:									