It is important to note that $sp(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ in $\mathbb{R}^n$ may not fill what we intuitively consider to be a $k$-dimensional portion of $\mathbb{R}^n$. For example, in $\mathbb{R}^2$ we see that $sp([1, -2], [-3, 6])$ is just the one-dimensional line along $[1, -2]$ because $[-3, 6] = -3[1, -2]$ already lies in $sp([1, -2])$. Similarly, if $\mathbf{v}_3$ is a vector in $sp(\mathbf{v}_1, \mathbf{v}_2)$, then $sp(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = sp(\mathbf{v}_1, \mathbf{v}_2)$ and so $sp(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is not three-dimensional. Section 2.1 will deal with this kind of *dependency* among vectors. As a result of our work there, we will be able to define dimensionality.

## SUMMARY

1. *Euclidean n-space* $\mathbb{R}^n$ consists of all ordered $n$-tuples of real numbers. Each $n$-tuple $\mathbf{x}$ can be regarded as a *point* $(x_1, x_2, \ldots, x_n)$ and represented graphically as a dot, or regarded as a vector $[x_1, x_2, \ldots, x_n]$ and represented by an arrow. The $n$-tuple $\mathbf{0} = [0, 0, \ldots, 0]$ is the *zero vector*. A real number $r \in \mathbb{R}$ is called a *scalar*.
2. Vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ can be added and subtracted, and each can be multiplied by a scalar $r \in \mathbb{R}$. In each case, the operation is performed on the components, and the resulting vector is again in $\mathbb{R}^n$. Properties of these operations are summarized in Theorem 1.1. Graphic interpretations are shown in Figures 1.6, 1.8, and 1.9.
3. Two nonzero vectors in $\mathbb{R}^n$ are *parallel* if one is a scalar multiple of the other.
4. A *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is a vector of the form $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where each $r_i$ is a scalar. The set of all such linear combinations is the *span* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $sp(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.
5. Every vector in $\mathbb{R}^n$ can be expressed uniquely as a linear combination of the *standard basis vectors* $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_i$ has 1 as its $i$th component and zeros for all other components.

## EXERCISES

*In Exercises 1–4, compute* $\mathbf{v} + \mathbf{w}$ *and* $\mathbf{v} - \mathbf{w}$ *for the given vectors* $\mathbf{v}$ *and* $\mathbf{w}$. *Then draw coordinate axes and sketch, using your answers, the vectors* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, *and* $\mathbf{v} - \mathbf{w}$.

1. $\mathbf{v} = [2, -1]$, $\mathbf{w} = [-3, -2]$
2. $\mathbf{v} = [1, 3]$, $\mathbf{w} = [-2, 5]$
3. $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$
4. $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{w} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$

*In Exercises 5–8, let* $\mathbf{u} = [-1, 3, -2]$, $\mathbf{v} = [4, 0, -1]$, *and* $\mathbf{w} = [-3, -1, 2]$. *Compute the indicated vector.*

5. $3\mathbf{u} - 2\mathbf{v}$
6. $\mathbf{u} + 2(\mathbf{v} - 4\mathbf{w})$
7. $\mathbf{u} + \mathbf{v} - \mathbf{w}$
8. $4(3\mathbf{u} + 2\mathbf{v} - 5\mathbf{w})$

*In Exercises 9–12, compute the given linear combination of* **u** = [1, 2, 1, 0], **v** = [−2, 0, 1, 6], *and* **w** = [3, −5, 1, −2].

**9.** **u** − 2**v** + 4**w**

**10.** 3**u** + **v** − **w**

**11.** 4**u** − 2**v** + 4**w**

**12.** −**u** + 5**v** + 3**w**

*In Exercises 13–16, reproduce on your paper those vectors in Figure 1.16 that appear in the exercise, and then draw an arrow representing each of the following linear combinations. All of the vectors are assumed to lie in the same plane. Use the technique illustrated in Figure 1.7 when all three vectors are involved.*

**13.** 2**u** + 3**v**

**14.** −3**u** + 2**w**

**15.** **u** + **v** + **w**

**16.** 2**u** − **v** + $\frac{1}{2}$**w**

*In Exercises 17–20, reproduce on your paper those vectors in Figure 1.17 that appear in the exercise, and then use the technique illustrated in Example 6 to estimate scalars r and s such that the given equation is true. All of the vectors are assumed to lie in the same plane.*

**17.** **x** = $r$**u** + $s$**v**

**18.** **y** = $r$**u** + $s$**v**

**19.** **u** = $r$**x** + $s$**v**

**20.** **y** = $r$**u** + $s$**x**

*In Exercises 21–30, find all scalars c, if any exist, such that the given statement is true. Try to do some of these problems without using pencil and paper.*

**21.** The vector [2, 6] is parallel to the vector [$c$, −3].

**22.** The vector [$c^2$, −4] is parallel to the vector [1, −2].

**23.** The vector [$c$, −$c$, 4] is parallel to the vector [−2, 2, 20].

**24.** The vector [$c^2$, $c^3$, $c^4$] is parallel to the vector [1, −2, 4] with the same direction.

**25.** The vector [13, −15] is a linear combination of the vectors [1, 5] and [3, $c$].

**26.** The vector [−1, $c$] is a linear combination of the vectors [−3, 5] and [6, −11].

**27.** **i** + $c$**j** − 3**k** is a linear combination of **i** + **j** and **j** + 3**k**.

**28.** **i** + $c$**j** + ($c$ − 1)**k** is in the span of **i** + 2**j** + **k** and 3**i** + 6**j** + 3**k**.

**29.** The vector 3**i** − 2**j** + $c$**k** is in the span of **i** + 2**j** − **k** and **j** + 3**k**.

**30.** The vector [$c$, −2$c$, $c$] is in the span of [1, −1, 1], [0, 1, −3], and [0, 0, 1].

*In Exercises 31–34, find the vector which, when translated, represents geometrically an arrow reaching from the first point to the second.*

**31.** From (−1, 3) to (4, 2) in $\mathbb{R}^2$

**32.** From (−3, 2, 5) to (4, −2, −6) in $\mathbb{R}^3$

**33.** From (2, 1, 5, −6) to (3, −2, 1, 7) in $\mathbb{R}^4$

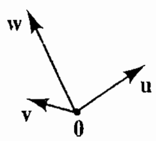


FIGURE 1.16

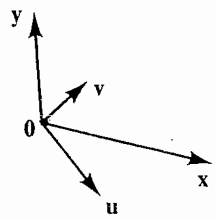


FIGURE 1.17

34. From (1, 2, 3, 4, 5) to (−5, −4, −3, −2, −1) in $\mathbb{R}^5$

35. Write the linear system

$$\begin{aligned} 3x - 2y + 4z &= 10 \\ x - y - 3z &= 0 \\ 2x + y - 5z &= -3 \end{aligned}$$

as a column-vector equation.

36. Write the linear system

$$\begin{aligned} x_1 - 3x_2 + 2x_3 \qquad &= -6 \\ 3x_1 \qquad - 4x_3 + 5x_4 &= 12 \end{aligned}$$

as a column-vector equation.

37. Write the row-vector equation

$$p[-3, 4, 6] + q[0, -2, 5] - r[4, -3, 2] + s[6, 0, 7] = [8, -3, 1]$$

as
a. a linear system,
b. a column-vector equation.

38. Write the column-vector equation

$$r_1\begin{bmatrix}-2\\3\\0\end{bmatrix} + r_2\begin{bmatrix}5\\13\\-4\end{bmatrix} + r_3\begin{bmatrix}16\\0\\-9\end{bmatrix} = \begin{bmatrix}5\\-8\\11\end{bmatrix}$$

as a linear system.

39. Mark each of the following True or False.
___ a. The notion of a vector in $\mathbb{R}^n$ is useful only if $n = 1, 2,$ or $3$.
___ b. Every ordered $n$-tuple in $\mathbb{R}^n$ can be viewed both as a point and as a vector.
___ c. It would be impossible to define addition of points in $\mathbb{R}^n$ because we only add vectors.
___ d. If **a** and **b** are two vectors in standard position in $\mathbb{R}^n$, then the arrow from the tip of **a** to the tip of **b** is a translated representation of the vector **a** − **b**.
___ e. If **a** and **b** are two vectors in standard position in $\mathbb{R}^n$, then the arrow from the tip of **a** to the tip of **b** is a translated representation of the vector **b** − **a**.
___ f. The span of any two nonzero vectors in $\mathbb{R}^2$ is all of $\mathbb{R}^2$.
___ g. The span of any two nonzero, nonparallel vectors in $\mathbb{R}^2$ is all of $\mathbb{R}^2$.
___ h. The span of any three nonzero, nonparallel vectors in $\mathbb{R}^3$ is all of $\mathbb{R}^3$.
___ i. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are vectors in $\mathbb{R}^2$ such that $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \mathbb{R}^2$, then $k = 2$.
___ j. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are vectors in $\mathbb{R}^3$ such that $\mathrm{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \mathbb{R}^3$, then $k \geq 3$.

40. Prove the indicated property of vector addition in $\mathbb{R}^n$, stated in Theorem 1.1.
a. Property A1
b. Property A3
c. Property A4

41. Prove the indicated property of scalar multiplication in $\mathbb{R}^n$, stated in Theorem 1.1.
a. Property S1
b. Property S3
c. Property S4

42. Prove algebraically that the linear system

$$\begin{aligned} r - 2s &= b_1 \\ 3r + 5s &= b_2 \end{aligned}$$

has a solution for all numbers $b_1, b_2 \in \mathbb{R}$, as asserted in the text.

43. Option 1 of the routine VECTGRPH in the software package LINTEK gives graphic quizzes on addition and subtraction of vectors in $\mathbb{R}^2$. Work with this Option 1 until you consistently achieve a score of 80% or better on the quizzes.

44. Repeat Exercise 43 using Option 2 of the routine VECTGRPH. The quizzes this time are on linear combinations in $\mathbb{R}^2$, and are quite similar to Exercises 17–20.

## MATLAB

The MATLAB exercises are designed to build some familiarity with this widely used software as you work your way through the text. Complete information can be obtained from the manual that accompanies MATLAB. Some information is