SUMMARY

1. An $m \times n$ matrix is an ordered rectangular array of numbers containing m rows and n columns.
2. An $m \times 1$ matrix is a column vector with m components, and a $1 \times n$ matrix is a row vector with n components.
3. The product $A\mathbf{b}$ of an $m \times n$ matrix A and a column vector $\mathbf{b}$ with components $b_1, b_2, \ldots, b_n$ is the column vector equal to the linear combination of the column vectors of A where the scalar coefficient of the jth column vector of A is b_j.
4. The product AB of an $m \times n$ matrix A and an $n \times s$ matrix B is the $m \times s$ matrix C whose jth column is A times the jth column of B. The entry c_{ij} in the ith row and jth column of C is the dot product of the ith row vector of A and the jth column vector of B. In general, $AB \neq BA$.
5. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of the same size, then $A + B$ is the matrix of that size with entry $a_{ij} + b_{ij}$ in the ith row and jth column.
6. For any matrix A and scalar r, the matrix rA is found by multiplying each entry in A by r.
7. The transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T, which has as its kth row vector the kth column vector of A.
8. Properties of the matrix operations are given in boxed displays on page 45.

EXERCISES

In Exercises 1–16, let

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \\ 0 & 6 \\ -3 & 2 \end{bmatrix}, \quad \textit{and} \quad D = \begin{bmatrix} -4 & 2 \\ 3 & 5 \\ -1 & -3 \end{bmatrix}.$$

Compute the indicated quantity, if it is defined.

1. $3A$
2. $0B$
3. $A + B$
4. $B + C$
5. $C - D$
6. $4A - 2B$
7. AB
8. $(CD)^T$
9. $(2A)(5C)$
10. $(5D)(4B)$
11. A^2
12. $(AC)^2$
13. $(2A - B)D$
14. ADB
15. $(A^T)A$
16. BC and CB

17. Let
$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
   a. Find A^2.
   b. Find A^7.

18. Let
$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$
   a. Find A^2.
   b. Find A^7.

19. Consider the row and column vectors

$$\mathbf{x} = [-2, 3, -1] \text{ and } \mathbf{y} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}.$$

Compute the matrix products **xy** and **yx**.

20. Fill in the missing entries in the 4×4 matrix

$$\begin{bmatrix} 1 & -1 & & 5 \\ & 4 & & 8 \\ 2 & -7 & -1 & \\ & & 6 & 3 \end{bmatrix}$$

so that the matrix is symmetric.

21. Mark each of the following True or False. The statements involve matrices A, B, and C that are assumed to have appropriate size.

___ a. If $A = B$, then $AC = BC$.
___ b. If $AC = BC$, then $A = B$.
___ c. If $AB = O$, then $A = O$ or $B = O$.
___ d. If $A + C = B + C$, then $A = B$.
___ e. If $A^2 = I$, then $A = \pm I$.
___ f. If $B = A^2$ and if A is $n \times n$ and symmetric, then $b_{ii} \geq 0$ for $i = 1, 2, \ldots, n$.
___ g. If $AB = C$ and if two of the matrices are square, then so is the third.
___ h. If $AB = C$ and if C is a column vector, then so is B.
___ i. If $A^2 = I$, then $A^n = I$ for all integers $n \geq 2$.
___ j. If $A^2 = I$, then $A^n = I$ for all even integers $n \geq 2$.

22. a. Prove that, if A is a matrix and **x** is a row vector, then **x**A (if defined) is again a row vector.
b. Prove that, if A is a matrix and **y** is a column vector, then A**y** (if defined) is again a column vector.

23. Let A be an $m \times n$ matrix and let **b** and **c** be column vectors with n components. Express the dot product $(A\mathbf{b}) \cdot (A\mathbf{c})$ as a product of matrices.

24. The product $A\mathbf{b}$ of a matrix and a column vector is equal to a linear combination of columns of A where the scalar coefficient of the jth column of A is b_j. In a similar fashion, describe the product $\mathbf{c}A$ of a row vector **c** and a matrix A as a linear combination of vectors. [HINT: Consider $((\mathbf{c}A)^T)^T$.]

In Exercises 25–34, prove that the given relation holds for all vectors, matrices, and scalars for which the expressions are defined.

25. $A + B = B + A$
26. $(A + B) + C = A + (B + C)$
27. $(r + s)A = rA + sA$
28. $(rs)A = r(sA)$
29. $A(B + C) = AB + AC$
30. $(A^T)^T = A$
31. $(A + B)^T = A^T + B^T$
32. $(AB)^T = B^TA^T$
33. $(AB)C = A(BC)$
34. $(rA)B = A(rB) = r(AB)$

35. If B is an $m \times n$ matrix and if $B = A^T$, find the size of
a. A,
b. AA^T,
c. A^TA.

36. Let **v** and **w** be column vectors in $\mathbb{R}^n$. What is the size of $\mathbf{v}\mathbf{w}^T$? What relationships hold between $\mathbf{v}\mathbf{w}^T$ and $\mathbf{w}\mathbf{v}^T$?

37. The Hilbert matrix H_n is the $n \times n$ matrix $[h_{ij}]$, where $h_{ij} = 1/(i + j - 1)$. Prove that the matrix H_n is symmetric.

38. Prove that, if A is a square matrix, then the matrix $A + A^T$ is symmetric.

39. Prove that, if A is a matrix, then the matrix AA^T is symmetric.

40. a. Prove that, if A is a square matrix, then $(A^2)^T = (A^T)^2$ and $(A^3)^T = (A^T)^3$. [HINT: Don't try to show that the matrices have equal entries; instead use Exercise 32.]
b. State the generalization of part (a), and give a proof using mathematical induction (see Appendix A).

41. a. Let A be an $m \times n$ matrix, and let $\mathbf{e}_j$ be the $n \times 1$ column vector whose jth component is 1 and whose other components are 0. Show that $A\mathbf{e}_j$ is the jth column vector of A.

b. Let A and B be matrices of the same size.

i. Prove that, if $A\mathbf{x} = \mathbf{0}$ (the zero vector) for all $\mathbf{x}$, then $A = O$, the zero matrix. [HINT: Use part (a).]

ii. Prove that, if $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x}$, then $A = B$. [HINT: Consider $A - B$.]

42. Let A and B be square matrices. Is

$$(A + B)^2 = A^2 + 2AB + B^2?$$

If so, prove it; if not, give a counterexample and state under what conditions the equation is true.

43. Let A and B be square matrices. Is

$$(A + B)(A - B) = A^2 - B^2?$$

If so, prove it; if not, give a counterexample and state under what conditions the equation is true.

44. An $n \times n$ matrix C is **skew symmetric** if $C^T = -C$. Prove that every square matrix A can be written *uniquely* as $A = B + C$ where B is symmetric and C is skew symmetric.

Matrix A **commutes** *with matrix B if $AB = BA$.*

45. Find all values of r for which

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix} \quad \text{commutes with} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

46. Find all values of r for which

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{commutes with} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The software LINTEK includes a routine, MATCOMP, that performs the matrix operations described in this section. Let

$$A = \begin{bmatrix} 4 & 6 & 0 & 1 & -9 \\ 2 & 11 & 5 & 2 & -5 \\ -1 & 2 & -4 & 5 & 7 \\ 0 & 12 & -8 & 4 & 3 \\ 10 & 4 & 6 & 2 & -5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -8 & 15 & 4 & -11 \\ 3 & 5 & 6 & -2 \\ 0 & -1 & 12 & 5 \\ 1 & 13 & -15 & 7 \\ 6 & -8 & 0 & -5 \end{bmatrix}.$$

Use MATCOMP in LINTEK to enter and store these matrices, and then compute the matrices in Exercises 47–54, if they are defined. Write down to hand in, if requested, the entry in the 3rd row, 4th column of the matrix.

47. $A^4 + A$

48. A^2B

49. $A^3(A^T)^2$

50. BA^2

51. $B^T(2A)$

52. $AB(AB)^T$

53. $(2A)^3 - A^5$

54. $(A^T)^5$

MATLAB

To enter a matrix in MATLAB, start with a left bracket **[** and then type the entries across the rows, separating the entries by spaces and separating the rows by semicolons. Conclude with a right bracket **]**. To illustrate, we would enter the matrix

$$A = \begin{bmatrix} -1 & 5 \\ 13 & -4 \\ 7 & 0 \end{bmatrix} \quad \text{as} \quad$$ **A = [−1 5; 13 −4; 7 0]**

and MATLAB would then print it for us to proofread. Recall that to avoid having data printed again on the screen, we type a semicolon at the end of the data before pressing the Enter key. Thus if we enter

A = [−1 5; 13 −4; 7 0];