Section 4.3 Computation of Determinants and Creamer's Rule

11.

$$\text{Let } A = \left[\begin{array}{c|c} R & O \\ \hline O & S \end{array}\right]$$

where A is an $n \times n$ matrix with an $r \times r$ submatrix R and an $s \times s$ submatrix S. $(n = r + s)$

Let prove the exercises 11 by induction on r.

When $r = 1$, if we expand $\det(A)$ by minors in the top row, then obviously the $\det(A) = \det(R) \cdot \det(S)$ holds.

Assume that $r > 1$ and the result holds if R is $(r-1) \times (r-1)$.

Now we have A is an $n \times n$ matrix with an $r \times r$ submatrix R and an $s \times s$ submatrix S. Let $A = [a_{i,j}]$ and $R = [r_{i,j}]$. Notice that

$$a_{1,j} = \begin{cases} r_{1,j}, & \text{for } j \le r \\ 0, & \text{for } r+1 \le j \le n \end{cases}$$

We expand $\det(A)$ by minors in the top row, then

$$\begin{aligned} \det(A) &= a_{1,1}a'_{1,1} + a_{1,2}a'_{1,2} + \dots + a_{1,r}a'_{1,r} + a_{1,r+1}a'_{1,r+1} + \dots + a_{1,n}a'_{1,n} \\ &= r_{1,1}a'_{1,1} + r_{1,2}a'_{1,2} + \dots + r_{1,r}a'_{1,r} + 0 \cdot a'_{1,r+1} + \dots + 0 \cdot a'_{1,n} \\ &= r_{1,1}a'_{1,1} + r_{1,2}a'_{1,2} + \dots + r_{1,r}a'_{1,r} \end{aligned}$$

For $j \le r$, each cofactor $a'_{1,j}$ is

$$a'_{1,j} = (-1)^{1+j} \det(A_{1,j}) = (-1)^{1+j} \left[\begin{array}{c|c} R_{1,j} & O \\ \hline O & S \end{array}\right]$$

where $A_{1,j}$ is the corresponding minor matrix of A and $R_{1,j}$ is the corresponding minor matrix of R. By the assumption of induction, we have

$$\det(A_{1,j}) = \det(R_{1,j}) \det(S)$$

Thus

$$\begin{aligned} &\det(A) = r_{1,1}a'_{1,1} + r_{1,2}a'_{1,2} + \dots + r_{1,r}a'_{1,r} \\ &= (-1)^{1+1} r_{1,1} \det(R_{1,1}) \det(S) + (-1)^{1+2} r_{1,2} \det(R_{1,2}) \det(S) + \dots + (-1)^{1+r} r_{1,r} \det(R_{1,r}) \det(S) \\ &= \left[(-1)^{1+1} r_{1,1} \det(R_{1,1}) + (-1)^{1+2} r_{1,2} \det(R_{1,2}) + \dots + (-1)^{1+r} r_{1,r} \det(R_{1,r})\right] \det(S) \\ &= \det(R) \det(S) \end{aligned}$$

22. Since $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A = (A^{-1})^{-1}$, we have

$$A = (A^{-1})^{-1} = \frac{1}{\det(A)} adj(A^{-1}) = \frac{1}{3} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

38. Let A be an $n \times n$ matrix. Prove that $\det(adj(A)) = \det(A)^{n-1}$.

Answer: By **Theorem 4.6** : $adj(A)A = (\det(A))I$.

$$\begin{aligned} adj(A)A &= (\det(A))I \\ \det(\ adj(A)A\ ) &= \det(\ (\det(A))I\ ) \\ \det(adj(A))\det(A) &= \det(A)^n \quad \text{Note that } I \text{ is an } n \times n \text{ matrix} \end{aligned}$$

If $\det(A) \neq 0$, it is easily to get $\det(adj(A)) = \det(A)^{n-1}$.

If $\det(A) = 0$, A is singular. By **Exercise 37**, $adj(A)$ is also singular.

$$\det(adj(A)) = \det(A)^{n-1} = 0.$$