Section 5-2 Diagonalization

17. Prove that, for every square matrix A all of whose eigenvalues are real, the product of its eigenvalues is $\det(A)$

Answer: If the characteristic polynomial of A is $p(\lambda) = |A - \lambda I|$, then $p(0) = |A| = \det(A)$. Also,

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

, so

$$p(0) = (-1)^{2n}\lambda_1\lambda_2\cdots\lambda_n = \lambda_1\lambda_2\cdots\lambda_n = \det(A).$$

18. Prove that similar square matrices have the same eigenvalues with the same algebraic multiplicities.

Answer: Let A and B are similar and $B = C^{-1}AC$. Then

$$\begin{aligned}\det(B - \lambda I) &= \det(C^{-1}AC - \lambda I) = \det(C^{-1}AC - C^{-1}(\lambda I)C)\\ &= \det(C^{-1}(A - \lambda I)C) = \det(C^{-1})\det(A - \lambda I)\det(C)\\ &= \det(A - \lambda I)\end{aligned}$$

Thus we know that A and B have the same characteristic polynomial. Therefore they have the same roots with the same multiplicities.

19. (a) Prove that if A is similar to rA where r is a real scalar other than 1 or -1, then all eigenvalues of A are zero. [*Hint:* 5-2 prob. 18.]

(b) What can you say about A if it is diagonalizable and similar to rA for some r where $|r| \neq 1$?

(c) Find a nonzero 2×2 matrix A which is similar to rA for every $r \neq 0$.

(d) Show that $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ is similar to $-A$.(Observe that the eigenvalues of A are not all zero.)

Answer: (a) If $r = 0$, trivial case!

If $r \neq 0$ and $|r| > 1$. Let λ_1 is an eigenvalue (possible complex) of A of maximum magnitude and there exist $\vec{v}_1 \neq \vec{0}$ so that $A\vec{v}_1 = \lambda_1\vec{v}_1$. Thus $(rA)\vec{v}_1 = (r\lambda_1)\vec{v}_1$ and $r\lambda_1$ is an eigenvalue of rA. Since A is similar to rA and use the idea of 5-2 prob. 18, we know that $r\lambda_1$ is also an eigenvalue of A. However, $|r\lambda_1| > |\lambda_1|$. $(\Rightarrow\Leftarrow)$

If $r \neq 0$ and $|r| < 1$. Let $\tilde{\lambda}_1$ is an eigenvalue (possible complex) of A of minimum magnitude. Similarly, we have $|r\tilde{\lambda}_1| < |\tilde{\lambda}_1|$. $(\Rightarrow\Leftarrow)$

(b) $A = O_{n\times n}$.

(c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(d) Need an invertible $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so that $C^{-1}AC = -A$, that is $AC = C(-A)$.

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} a+c & b+d \\ -c & -d \end{bmatrix} = \begin{bmatrix} -a & -a+b \\ -c & -c+d \end{bmatrix}$$

It is easy to check $a = 1,\ b = 0,\ c = -2,\ d = -1$ is a solution.

22. Let A and Cbe $n \times n$ matrices, and let C be invertible. Prove that, if $\vec{v}$ is an eigenvector of A with corresponding eigenvalues λ, then $C^{-1}\vec{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalues λ. Then prove that all eigenvectors of $C^{-1}AC$ are form $C^{-1}\vec{v}$, where $\vec{v}$ is an eigenvector of A.

Answer: Let $A\vec{v} = \lambda\vec{v}$. Then

$$(C^{-1}AC)(C^{-1}\vec{v}) = C^{-1}A(CC^{-1})\vec{v} = C^{-1}(A\vec{v}) = C^{-1}(\lambda\vec{v}) = \lambda(C^{-1}\vec{v})$$

Therefore, $C^{-1}\vec{v}$ is an eigenvector of $C^{-1}AC$ with corresponding eigenvalues λ.

Given an eigenvector $\vec{u}$ of $C^{-1}AC$ with corresponding eigenvalue α so that $C^{-1}AC\vec{u} = \alpha\vec{u}$. Then

$$A(C\vec{u}) = (CC^{-1})AC\vec{u} = C(C^{-1}AC\vec{u}) = C\alpha\vec{u} = \alpha C\vec{u}$$

Hence we know $C\vec{u}$ is an eigenvector of A with corresponding eigenvalue α. Thus $\vec{u} = C^{-1}(C\vec{u})$ has the requested form.