Section 6.1 Projections

25. Let W be a subspace of $\mathbb{R}^n$ and let $\vec{b}$ be a vector in $\mathbb{R}^n$. Prove that there is one and only one vector $\vec{p}$ in W such that $\vec{b}-\vec{p}$ is perpendicular to every vector in W . [HINT: Suppose that $\vec{p}_1$ and $\vec{p}_2$ are two such vectors, and show that $\vec{p}_1 - \vec{p}_2$ is in $W^\perp$.

Answer: Assume there're two vectors $\vec{p}_1, \vec{p}_2 \in W$ such that $\vec{b} - \vec{p}_1$ and $\vec{b} - \vec{p}_2$ are both perpendicular to every vector in W . i.e. $\vec{b} - \vec{p}_1$ and $\vec{b} - \vec{p}_2$ are both in $W^\perp$.

For all vector $\vec{v} \in W$

$$0 = \vec{v} \cdot (\vec{b} - \vec{p}_1) = \vec{v} \cdot \vec{b} - \vec{v} \cdot \vec{p}_1 \therefore \vec{v} \cdot \vec{b} = \vec{v} \cdot \vec{p}_1$$

$$0 = \vec{v} \cdot (\vec{b} - \vec{p}_2) = \vec{v} \cdot \vec{b} - \vec{v} \cdot \vec{p}_2 \therefore \vec{v} \cdot \vec{b} = \vec{v} \cdot \vec{p}_2$$

$$\therefore \vec{v} \cdot (\vec{p}_1 - \vec{p}_2) = 0$$

$$\therefore \vec{p}_1 - \vec{p}_2 \in W^\perp$$

Note that W is a vector space and $\vec{p}_1, \vec{p}_2 \in W$, we will have $\vec{p}_1 - \vec{p}_2 \in W^\perp$. Since $\vec{p}_1 - \vec{p}_2$ in both W and $W^\perp$, we can easily checked that $\vec{p}_1 - \vec{p}_2 = \vec{0}$.