Section 6.2 The Gram-Schmidt Process

24. Let B be the ordered orthonormal basis $\left(\vec{b}_1 = [\frac{1}{3}, \frac{2}{3}, \frac{-2}{3}], \vec{b}_2 = [\frac{2}{3}, \frac{1}{3}, \frac{2}{3}], \vec{b}_3 = [\frac{-2}{3}, \frac{2}{3}, \frac{1}{3}]\right)$ for $\mathbb{R}^3$

(a) Find the coordinate vectors $[c_1, c_2, c_3]$ for $[1, 2, -4]$ and $[d_1, d_2, d_3]$ for $[5, -3, 2]$, relative to the ordered basis B.

(b) Compute $[1, 2, -4] \cdot [5, -3, 2]$, and then compute $[c_1, c_2, c_3] \cdot [d_1, d_2, d_3]$. What do you notice?

Answer: (a) Let

$$A = \begin{bmatrix} \vec{b}_1^T & \vec{b}_2^T & \vec{b}_3^T \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 13/3 \\ -4/3 \\ -2/3 \end{bmatrix}$$

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -5/3 \\ 11/3 \\ -14/3 \end{bmatrix}$$

(b)

$$[1, 2, -4] \cdot [5, -3, 2] = -9$$

$$[13/3, -4/3, -2/3] \cdot [-5/3, 11/3, -14/3] = -9$$

Noticed that the results of inner product are the SAME, which should known by Theorem 6.6 property 1.

28. Find the QR-factorization of the matrix having as column vecotrs the transpose of the given row vectors from exercise 11.

Exercise 11: find the orthonormal basis for $sp([1, 0, 1, 0], [1, 1, 1, 0], [1, -1, 0, 1])$ of $\mathbb{R}^4$.

Answer: By Gram-Schmidt process. Let

$$\vec{a}_1 = [1, 0, 1, 0], \vec{a}_2 = [1, 1, 1, 0], \vec{a}_3 = [1, -1, 0, 1] \tag{1}$$

$$\vec{v}_1 = \vec{a}_1 = [1, 0, 1, 0], \vec{q}_1 = \frac{\vec{v}_1}{|\vec{v}_1|} = [\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0] \tag{2}$$

$$\vec{v}_2 = \vec{a}_2 - (\vec{a}_2 \cdot \vec{q}_1)\vec{q}_1 = \vec{a}_2 - \sqrt{2}\vec{q}_1 = [0, 1, 0, 0] \tag{3}$$

$$\vec{q}_2 = \frac{\vec{v}_2}{|\vec{v}_2|} = [0, 1, 0, 0] \tag{4}$$

$$\vec{v}_3 = \vec{a}_3 - (\vec{a}_3 \cdot \vec{q}_1)\vec{q}_1 - (\vec{a}_3 \cdot \vec{q}_2)\vec{q}_2 = \vec{a}_3 - \frac{\sqrt{2}}{2}\vec{q}_1 - \vec{q}_2 = [\frac{1}{2}, 0, \frac{-1}{2}, 1] \tag{5}$$

$$\vec{q}_3 = \frac{\vec{v}_3}{|\vec{v}_3|} = [\frac{1}{\sqrt{6}}, 0, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}] \tag{6}$$

By (1)

$$\vec{a}_1 = \sqrt{2}\,\vec{q}_1 \Rightarrow \begin{bmatrix}\vec{a}_1^T\end{bmatrix} = \begin{bmatrix}\vec{q}_1^T\end{bmatrix}\begin{bmatrix}\sqrt{2}\end{bmatrix} \tag{7}$$

By (3) and (4)

$$\vec{a}_2 = \sqrt{2}\,\vec{q}_1 + \vec{v}_2 = \sqrt{2}\,\vec{q}_1 + \vec{q}_2 \Rightarrow \begin{bmatrix}\vec{a}_2^T\end{bmatrix} = \begin{bmatrix}\vec{q}_1^T & \vec{q}_2^T\end{bmatrix}\begin{bmatrix}\sqrt{2}\\ 1\\ 0\end{bmatrix} \tag{8}$$

By (5) and (6)

$$\vec{a}_3 = \frac{\sqrt{2}}{2}\vec{q}_1 + \vec{q}_2 + \vec{v}_3 = \frac{\sqrt{2}}{2}\vec{q}_1 + \vec{q}_2 + \frac{\sqrt{3}}{\sqrt{2}}\,\vec{q}_3 \Rightarrow \begin{bmatrix}\vec{a}_3^T\end{bmatrix} = \begin{bmatrix}\vec{q}_1^T & \vec{q}_2^T & \vec{q}_3^T\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2}\\ 1\\ \frac{\sqrt{3}}{\sqrt{2}}\end{bmatrix} \tag{9}$$

Therefore,

$$\begin{bmatrix}\vec{a}_1^T & \vec{a}_2^T & \vec{a}_3^T\end{bmatrix} = \begin{bmatrix}\vec{q}_1^T & \vec{q}_2^T & \vec{q}_3^T\end{bmatrix}\begin{bmatrix}\sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2}\\ 0 & 1 & 1\\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{2}}\end{bmatrix}$$

That is

$$A = \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & -1\\ 1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = QR = \begin{bmatrix}1/\sqrt{2} & 0 & 1/\sqrt{6}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & -1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{6}\end{bmatrix}\begin{bmatrix}\sqrt{2} & \sqrt{2} & \frac{\sqrt{2}}{2}\\ 0 & 1 & 1\\ 0 & 0 & \frac{\sqrt{3}}{\sqrt{2}}\end{bmatrix}$$

31. Let A be an $n \times n$ matrix. Prove that the column vectors of A are orthogonal if and only if the row vectors of A are orthonormal. [*Hint:* Use Exercise 30 and the fact that A commutes with its inverse.]

Exercise 30: Let A be an $n \times n$ matrix. Prove that A has orthonormal column vectors if and only if A is invertible with inverse $A^{-1} = A^T$.

Answer: We have :

[A has orthonormal column vectors.]

iff [A is invertible with inverse $A^{-1} = A^T$.] (by Exercise 30)

iff [A^T is invertible with inverse $(A^T)^{-1} = (A^T)^T = A$.]

iff [A^T has orthonormal column vectors.] (by Exercise 30)

iff [A has orthonormal row vectors.]