Section 6.3 Orthogonal Matrices

7. A is a square matrix below, please find A^{-1} by the given method.

$$A = \begin{bmatrix} 4 & -3 & 6 \\ 6 & 6 & 2 \\ -12 & 2 & 3 \end{bmatrix}$$

Method: If A and D are square matrices, D is diagonal, and AD is orthogonal, then $A^{-1} = D^2 A^T$

Answer: Name the column vectors of A are $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Notice that $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is orthogonal and $\|\vec{a}_1\| = 14, \|\vec{a}_2\| = 7, \|\vec{a}_3\| = 7$, hence $\left\{\frac{\vec{a}_1}{14}, \frac{\vec{a}_2}{7}, \frac{\vec{a}_3}{7}\right\}$ is orthonormal.

$$AD = \begin{bmatrix} 4 & -3 & 6 \\ 6 & 6 & 2 \\ -12 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1/14 & 0 & 0 \\ 0 & 1/7 & 0 \\ 0 & 0 & 1/7 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 & -3 & 6 \\ 3 & 6 & 2 \\ -6 & 2 & 3 \end{bmatrix} \text{ is an orthogonal matrix.}$$

$$A^{-1} = D^2 A^T \begin{bmatrix} 1/196 & 0 & 0 \\ 0 & 1/49 & 0 \\ 0 & 0 & 1/49 \end{bmatrix} \begin{bmatrix} 4 & 6 & -12 \\ -3 & 6 & 2 \\ 6 & 2 & 3 \end{bmatrix} = \frac{1}{49} \begin{bmatrix} 1 & 3/2 & -3 \\ -3 & 6 & 2 \\ 6 & 2 & 3 \end{bmatrix}$$

21. Let A be an orthogonal matrix. Show that A^2 is an orthogonal matrix, too.

Answer: A be an orthogonal matrix, i.e. $A^T A = I$. $(A^2)^T = (AA)^T = A^T A^T = (A^T)^2$. Therefore $(A^2)^T (A^2) = A^T A^T A A = A^T (A^T A) A = A^T I A = A^T A = I$. We have A^2 is an orthogonal matrix.

23. Find a 2×2 matrix with determinant 1 that is not an orthogonal matrix.

Answer:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\det(A) = 1$, but $A^T A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \neq I$

31. Let A and C be orthogonal $n \times n$ matrices. Show that CAC^{-1} is orthogonal.

Answer: A and C be orthogonal matrices, i.e. $A^TA = C^TC = I$. We also know that $C^{-1} = C^T$, i.e. $CAC^{-1} = CAC^T$

$(CAC^T)^T(CAC^T) = (C^T)^TA^TC^TCAC^T = CA^TC^TCAC^T = CA^T(C^TC)AC^T$

$= CA^TIAC^T = C(A^TA)C^T = CIC^T = (C^TC)^T = I^T = I.$

We have $CAC^T = CAC^{-1}$ is orthogonal.