Section 6.4 The Projection Matrix

20. Show that every symmetric matrix whose only eigenvalues are 0 and 1 is a projection matrix.

Note: The textbook discusses only real matrices in this section. Therefore, we add one more condition: the matrix is real.

Answer: Let A be a symmetric matrix whose only eigenvalues are 0 and 1. By Theorem 6.8, A is orthogonally diagonalizable; that is, there exists an orthogonal matrix C and a diagonal matrix D such that

$$C^{-1}AC = D.$$

By Theorem 6.12, to show that A is a projection matrix, it suffices to verify that A is both idempotent and symmetric.

Note that D has only 0 and 1 on its diagonal; hence $D^2 = D$. It follows that

$$A^2 = (C^{-1}DC)^2 = C^{-1}D^2C = C^{-1}DC = A.$$