# Section 9.1 Algebra of Complex Numbers

**5.** Show that $z$ is a real number if and only if $z = \overline{z}$.

**Answer:**
1. If $z$ is a real number of course $z = \overline{z}$.

2. Let $z = a + bi$, where $a, b \in \mathbb{R}$. If $z = \overline{z}$, we have $a + bi = z = \overline{z} = \overline{a + bi} = a - bi$. That is $b = -b$, we have $b = 0$. Therefore, $z = a \in \mathbb{R}$.

**15.** Illustrate Eqs. (5) in text for $z_1 = 2 + 2i$ and $z_2 = 1 + \sqrt{3}i$

**Geometric Representation of $z_1/z_2$**
1. $|z_1/z_2| = |z_1|/|z_2|$.
2. $Arg(z_1) - Arg(z_2)$ is an argument of $z_1/z_2$ (5)

**Answer:**
Let $z_1 = 2+2i, z_2 = 1+\sqrt{3}i$. Then $|z_1| = \sqrt{4+4} = 2\sqrt{2}$ and $Arg(|z_1|) = \arctan(1) = \frac{\pi}{4}$, while $|z_2| = 2$ and $Arg(z_2) = \frac{\pi}{3}$.

$$\frac{z_1}{z_2} = \frac{2+2i}{1+\sqrt{3}i} = \frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i$$

$$\therefore |z_1/z_2| = \sqrt{\left(\frac{1+\sqrt{3}}{2}\right)^2 + \left(\frac{1-\sqrt{3}}{2}\right)^2} = \sqrt{2} = \frac{2\sqrt{2}}{2} = \frac{|z_1|}{|z_2|}$$

Notice that

$$\tan(Arg(z1/z_2)) = \frac{1-\sqrt{3}}{1+\sqrt{3}} = \tan(\frac{-\pi}{12}).$$

Wh also notice that $z_1/z_2$ lies in the $4^{th}$ quadrant. It's easily get

$$Arg(z1/z_2) = \frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{3} = Arg(z_1) - Arg(z_2).$$

**25.** Let $z, w \in \mathbb{C}$. Show that $|z + w| \le |z| + |w|$. [HINT: Remember that $\mathbf{C}$ is a real vector space of dimension 2, naturally isomorphic to $\mathbf{R}^2$]

**Answer:**
Let $z = a+bi, w = c+di$ in $\mathbf{C}$, and let $\vec{u} = [a, b], \vec{v} = [c, d]$ in $\mathbf{R}^2$. Since $\|\vec{u}\| = |z|, \|\vec{v}\| = |w|$, and $\|\vec{u} + \vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$ by the triangle inequality, we have $|z + w| \le |z| + |w|$.