9.2 Matrices ans Vector Spaces with Complex Scalars

35. Prove that an $n \times n$ matrix U unitary if and only if the rows of U form an orthonormal basis for $\mathbb{C}^n$.

Answer:

$\boxed{\Longrightarrow}$ If U is unitary, by the definition, its column vectors are orthogonal unit vectors $(\vec{c}_1, \vec{c}_2, ..., \vec{c}_n)$. Assume there's $r_1, r_2, ..., r_n \in \mathbb{C}$, such that $r_1\vec{c}_1 + ... + r_n\vec{c}_n = \vec{0}$.

$$\because \langle \vec{c}_1, r_1\vec{c}_1 + ... + r_n\vec{c}_n \rangle = \left\langle \vec{c}_1, \vec{0} \right\rangle = 0$$

$$\therefore r_1 \langle \vec{c}_1, \vec{c}_1 \rangle + ... + r_n \langle \vec{c}_1, \vec{c}_n \rangle = r_1 \langle \vec{c}_1, \vec{c}_1 \rangle = r_1 = 0$$

Similarly, $r_2 = ... = r_n = 0$. Therefore, $\vec{c}_1, \vec{c}_2, ..., \vec{c}_n$ are linearly independent, that is, the column vectors of U are an orthonormal basis.

Since $U^*U = I = UU^* = \overline{U}U^T = (U^T)^*U^T$, U^T is also an unitary matrix. That is, the column vectors of U^T is also an orthonormal basis. Therefore, the row vectors of U is an orthonormal basis.

$\boxed{\Longleftarrow}$ Let the row vectors of U is $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$.

$$\overline{U}U^T = \overline{\begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_n \end{bmatrix}} \begin{bmatrix} \vec{v}_1^T & \vec{v}_2^T & ... & \vec{v}_n^T \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I$$

$I^T = (\overline{U}U^T)^T = UU^*$. Hence U is unitary.

37. Prove that the product of two commuting $n \times n$ Hermitian matrices is also a Hermitian matrix. What can you say about the sum of two Hermitian matrices?

Answer:

Let H_1, H_2 are Hermitian matrices, i.e. $H_1^* = H_1, H_2^* = H_2$. Since H_11, H_2 are commuting, i.e. $H_1H_2 = H_2H_1$ $(H_1H_2)^* = H_2^*H_1^* = H_2H_1 = H_1H_2$. Hence H_1H_2 is a Hermitian matrix.

$(H_1 + H_2)^* = H_1^* + H_2^* = H_1 + H_2$. Hence $H_1 + H_2$ is a Hermitian matrix.

39. Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation whose standard matrix representation is a unitary matrix U. Show that $\langle T(\vec{u}), T(\vec{v}) \rangle = \langle \vec{u}, \vec{v} \rangle$, for all $\vec{u}, \vec{v} \in \mathbb{C}^n$

Answer:

I already proved in class.

40. Prove that for $\vec{u}, \vec{v} \in \mathbb{C}^n$, $(\vec{u}^*\vec{v})^* = \overline{\vec{u}^*\vec{v}} = \vec{v}^*\vec{u} = \vec{u}^T\overline{\vec{v}}$

Answer:

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$$\begin{aligned}
(\vec{u}^*\vec{v})^* &= (\begin{bmatrix} \overline{u_1} & \overline{u_2} & \dots & \overline{u_n} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix})* = (\sum_{i=1}^{n} \overline{u_i} v_i)^* \\
&= \overline{\sum_{i=1}^{n} \overline{u_i} v_i} (= \overline{\vec{u}^*\vec{v}}) \\
&= \sum_{i=1}^{n} u_i \overline{v_i} \\
&= \begin{bmatrix} \overline{v_1} & \overline{v_2} & \dots & \overline{v_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} (= \vec{v}^*\vec{u}) \\
&= \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix} \begin{bmatrix} \overline{v_1} \\ \overline{v_2} \\ \vdots \\ \overline{v_n} \end{bmatrix} (= \vec{u}^T\overline{\vec{v}})
\end{aligned}$$

43. A square matrix A is normal if $A^*A = AA^*$

(a) Show that every Hermitian matrix is normal.

(b) Show that every unitary matrix is normal.

(c) Show that, if $A^* = -A$, then A is normal.

Answer:

(a) Let H are Hermitian matrices, i.e. $H^* = H$. $HH^* = HH = H^*H$.

(b) Let U are unitary matrices, i.e. $U^*U = I$, i.e. $U^{-1} = U^*$. $UU^* = I + U^*U$.

(c) If $A^* = -A$, $A * A = (-A)A = -AA = A(-A) = AA^*$.

44. Let A be an $n \times n$ matrix. Referring to Exercise 43, prove that , if A is normal, then $\|A\vec{z}\| = \|A^*\vec{z}\|$ for all $\vec{z} \in \mathbb{C}^n$.

Answer:
$\|A\vec{z}\|^2 = (A\vec{z})^*(A\vec{z}) = \vec{z}^*A^*A\vec{z} = \vec{z}^*AA^*\vec{z} = (A^*\vec{z})^*(A^*\vec{z}) = \|A^*\vec{z}\|^2$.
Since all norms are real and positive, $\|A\vec{z}\| = \|A^*\vec{z}\|$.