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Homework 9-3

Section 9-3

Extra:

$$A = \begin{bmatrix} 4 & -4 & -3 \\ 1 & 2 & -1 \\ 2 & -4 & -1 \end{bmatrix}$$

Use the process in Schur's Lemma to find an unitary matrix U such that $U^{-1}AU$ is an upper triangular R.

(答案在下一頁)

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注意：這題答案的 U 跟 R 都不是唯一的。下面只是列出一種可能。

答案: (a)

$$p(A) = (1-\lambda)(2-\lambda)^2$$

Pick $\lambda = 1$

$$rref(A-I) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \ \Rightarrow \ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Extend $\vec{v}_1$ into a basis for $\mathbb{C}^3$, and by Gram-Schmidt process we can transform it into an orthonormal basis $\{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$. Let $U_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$.

(b)

$$U_1^* A U_1 = \begin{bmatrix} 1 & 5 & -4\sqrt{2} \\ 0 & 2 & 0 \\ 0 & \sqrt{2} & 2 \end{bmatrix} = \left[\begin{array}{c|cc} 1 & * & * \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{A}$}} \\ 0 & & \end{array}\right], \ \tilde{A} = \begin{bmatrix} 2 & 0 \\ \sqrt{2} & 2 \end{bmatrix}$$

$$|\tilde{A} - \lambda I| = (2-\lambda)^2$$

$$rref(\tilde{A} - 2I) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \vec{v}_1' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Extend $\vec{v}_1'$ into a basis for $\mathbb{C}^2$, and by Gram-Schmidt process we can transform it into an orthonormal basis $\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}$

$$\tilde{U} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } \tilde{U}^* \tilde{A} \tilde{U} = \begin{bmatrix} 2 & * \\ 0 & * \end{bmatrix}$$

$$U_2 = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\tilde{U}$}} \\ 0 & & \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) Combine (a) and (b).

$$U = U_1 U_2 = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \end{bmatrix}$$

Check:

$$U^* A U = U_2^* U_1^* A U_1 U_2 = \begin{bmatrix} 1 & -4\sqrt{2} & 5 \\ 0 & 2 & \sqrt{2} \\ 0 & 0 & 2 \end{bmatrix}$$