Section 9.3 Eigenvalues and Diagonalization

17. Prove that every 2×2 real matrix that is unitarily diagonalizable has one of the following forms $\begin{bmatrix} a & b \\ b & d \end{bmatrix}$, $\begin{bmatrix} a & b \\ -b & d \end{bmatrix}$ for $a, b, d \in \mathbb{R}$

Answer:

Every 2×2 real matrix A can written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since A is unitarily diagonalizable, A is normal, i.e. $A^*A = AA^*$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^*$$

$$\begin{bmatrix} a\overline{a} + c\overline{c} & \overline{a}b + \overline{c}d \\ a\overline{b} + c\overline{d} & b\overline{b} + d\overline{d} \end{bmatrix} = \begin{bmatrix} a\overline{a} + b\overline{b} & a\overline{c} + b\overline{d} \\ \overline{a}c + \overline{b}d & c\overline{c} + d\overline{d} \end{bmatrix}$$

Hence: (notice that a, b, c, d are real.)

(1). $a\overline{a} + c\overline{c} = a\overline{a} + b\overline{b} \Longrightarrow a^2 + c^2 = a^2 + b^2$

(2). $\overline{a}b + \overline{c}d = a\overline{c} + b\overline{d} \Longrightarrow ab + cd = ac + bd$

(3). $a\overline{b} + c\overline{d} = \overline{a}c + \overline{b}d \Longrightarrow ab + cd = ac + bd$

(4). $b\overline{b} + d\overline{d} = c\overline{c} + d\overline{d} \Longrightarrow b^2 + d^2 = c^2 + d^2$

by (1) and (4), we have $b = c$ or $b = -c$. And (2)(3) holds for for both cases.

23. Prove that an $n \times n$ matrix A is unitarily diagonalizable if and only if $\|A\vec{v}\| = \|A^*\vec{v}\|$ for all $\vec{v} \in \mathbb{C}^n$.

Answer:

If A is unitarily diagonalizable, then there exists an unitary U and a diagonal matrix D such that $A = UDU^*$.

$$\begin{aligned}\|A\vec{v}\|^2 &= (A\vec{v})^*(A\vec{v}) = \vec{v}^*A^*A\vec{v} = \vec{v}^*(UDU^*)^*(UDU^*)\vec{v} = \vec{v}^*UD^*U^*UDU^*\vec{v} \\ &= \vec{v}^*UD^*DU^*\vec{v} = \vec{v}^*UDD^*U^*\vec{v} = \vec{v}^*UDU^*UD^*U^*\vec{v} = \vec{v}^*(UD^*U^*)^*(UD^*U^*)\vec{v} \\ &= \|A^*\vec{v}\|^2\end{aligned}$$

Since all norms are real and positive, $\|A\vec{v}\| = \|A^*\vec{v}\|$.

If $\|A\vec{v}\| = \|A^*\vec{v}\|$, we have $\|A\vec{v}\|^2 = (A\vec{v})^*(A\vec{v}) = \vec{v}^*A^*A\vec{v} = \vec{v}^*AA^*\vec{v} = \|A^*\vec{v}\|^2$. That is $\vec{v}^*A^*A\vec{v} = \vec{v}^*AA^*\vec{v}$ for all $\vec{v} \in \mathbb{C}^n$. We have $A^*A = AA^*$. By theorem 9.7, A is unitarily diagonalizable.