Section 9.4 Jordan Canonical Form

19. Find a Jordan canonical form and a Jordan basis for the given matrix.

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Answer:

$$A = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & & \\ & \boxed{\begin{matrix} -1 & 1 \\ 0 & -1 \end{matrix}} & \\ & & \boxed{1} \end{bmatrix}$$

33. Let A be an $n \times n$ matrix with eigenvalue λ. Prove that the algebraic multiplicity of λ is at least as large as its geometric multiplicity.

Answer:

Assume the Jordan Canonical form of A contains k λ-Jordan blocks, which has size $m_1, m_2, ..., m_k$ and $m_1 + m_2 + ... + m_k = M$. Since the characteristic polynomial the Jordan Canonical form has factor $(x - \lambda)^M$ and the characteristic polynomial of A and its the Jordan Canonical form are the same, the algebraic multiplicity of λ is M.

The geometric multiplicity of λ is the dimension of its eigenspace. Therefore, the geometric multiplicity of λ is the number of λ-Jordan blocks, which is k.

Obviously, $k \leq M$.

$$A \sim J = \begin{bmatrix} J_{m_1}(\lambda) & & & & \\ & J_{m_2}(\lambda) & & & \\ & & \ddots & & \\ & & & J_{m_k}(\lambda) & \\ & & & & \text{other jordan blocks} \end{bmatrix}$$