CHAPTER

1 VECTORS, MATRICES, AND LINEAR SYSTEMS

We have all solved simultaneous linear equations—for example,

$$\begin{aligned} 2x + y &= 4 \\ x - 2y &= -3. \end{aligned}$$

We shall call any such collection of simultaneous linear equations a *linear system*. Finding all solutions of a linear system is fundamental to the study of linear algebra. Indeed, the great practical importance of linear algebra stems from the fact that *linear systems can be solved by algebraic methods.* For example, a *linear* equation in one unknown, such as $3x = 8$, is easy to solve. But the nonlinear equations $x^5 + 3x = 1$, $x^x = 100$, and $x - \sin x = 1$ are all difficult to solve algebraically.

One often-used technique for dealing with a nonlinear problem consists of *linearizing the problem*—that is, approximating the problem with a linear one that can be solved more easily. Linearization techniques often involve calculus. If you have studied calculus, you may be familiar with Newton's method for approximating a solution to an equation of the form $f(x) = 0$; an example would be $x - 1 - \sin x = 0$. An approximate solution is found by solving sequentially several linear equations of the form $ax = b$, which are obtained by approximating the graph of f with lines. Finding an approximate numerical solution of a partial differential equation may involve solving a linear system consisting of thousands of equations in thousands of unknowns. With the advent of the computer, solving such systems is now possible. The feasibility of solving huge linear problems makes linear algebra currently one of the most useful mathematical tools in both the physical and the social sciences.

The study of linear systems and their solutions is phrased in terms of *vectors* and *matrices*. Sections 1.1 and 1.2 introduce vectors in the Euclidean spaces (the plane, 3-space, etc.) and provide a geometric foundation for our work. Sections 1.3–1.6 introduce matrices and methods for solving linear systems and study solution sets of linear systems.

1.1 VECTORS IN EUCLIDEAN SPACES

We all know the practicality of two basic arithmetic operations—namely, adding two numbers and multiplying one number by another. We can regard the real numbers as forming a line which is a one-dimensional space. In this section, we will describe a useful way of adding two points in a plane, which is a two-dimensional space, or two points in three-dimensional space. We will even describe what is meant by n-dimensional space and define addition of two points there. We will also describe how to multiply a point in two-, three-, and n-dimensional space by a real number. These extended notions of addition and of multiplication by a real number are as useful in n-dimensional space for $n > 1$ as they are for the one-dimensional real number line. When these operations are performed in spaces of dimension greater than one, it is conventional to call the elements of the space *vectors* as well as *points*. In this section, we describe a physical model that suggests the term *vector* and that motivates addition of vectors and multiplication of a vector by a number. We then formally define these operations and list their properties.

Euclidean Spaces

Let $\mathbb{R}$ be the set of all real numbers. We can regard $\mathbb{R}$ geometrically as the Euclidean line—that is, as **Euclidean 1-space**. We are familiar with rectangular x,y-coordinates in the Euclidean plane. We consider each ordered pair (a, b) of real numbers to represent a point in the plane, as illustrated in Figure 1.1. The set of all such ordered pairs of real numbers is **Euclidean 2-space**, which we denote by $\mathbb{R}^2$, and often call *the plane.*

To coordinatize space, we choose three mutually perpendicular lines as coordinate axes through a point that we call the *origin* and label $\mathbf{0}$, as shown in Figure 1.2. Note that we represent only half of each coordinate axis for clarity. The coordinate system in this figure is called a *right-hand* system because, when the fingers of the right hand are curved in the direction required to rotate the positive x-axis toward the positive y-axis, the right thumb points up the z-axis, as shown in Figure 1.2. The set of all ordered triples (a, b, c) of real numbers is **Euclidean 3-space**, denoted $\mathbb{R}^3$, and often simply referred to as *space.*

Although a Euclidean space of dimension four or more may be difficult for us to visualize geometrically, we have no trouble writing down an ordered quadruple of real numbers such as $(2, -3, 7, \pi)$ or an ordered quintuple such as $(-0.3, 3, 2, -5, 21.3)$, etc. Indeed, it can be useful to do this. A household budget might contain nine categories, and the expenses allowed per week in each category could be represented by an ordered 9-tuple of real numbers. Generalizing, the set $\mathbb{R}^n$ of all ordered n-tuples $(x_1, x_2, \ldots, x_n)$ of real numbers is **Euclidean n-space**. Note the use of just one letter with consecutive integer subscripts in this n-tuple, rather than different letters. We will often denote an element of $\mathbb{R}^2$ by (x_1, x_2) and an element of $\mathbb{R}^3$ by (x_1, x_2, x_3).

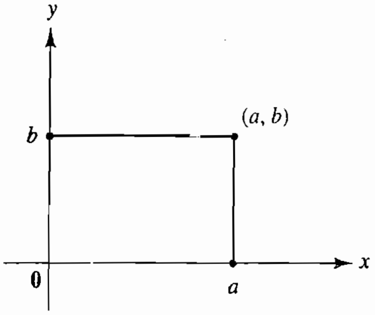

FIGURE 1.1
Rectangular coordinates in the plane.

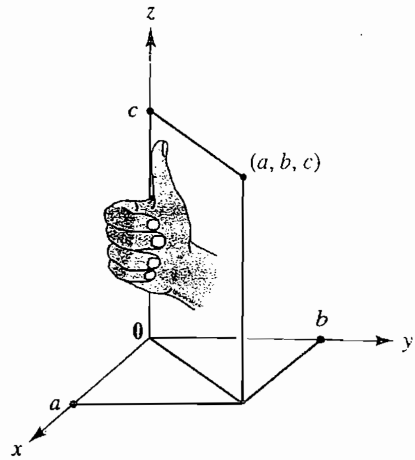

FIGURE 1.2
Rectangular coordinates in space.

The Physical Notion of a Vector

We are accustomed to visualizing an ordered pair or triple as a point in the plane or in space and denoting it geometrically by a dot, as shown in Figures 1.1 and 1.2. Physicists have found another very useful geometric interpretation of such pairs and triples in their consideration of forces acting on a body. The motion in response to a force depends on the *direction* in which the force is applied and on the *magnitude* of the force—that is, on how hard the force is exerted. It is natural to represent a force by an arrow, pointing in the direction

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HISTORICAL NOTE THE IDEA OF AN n-DIMENSIONAL SPACE FOR $n > 3$ reached acceptance gradually during the nineteenth century; it is thus difficult to pinpoint a first "invention" of this concept. Among the various early uses of this notion are its appearances in a work on the divergence theorem by the Russian mathematician Mikhail Ostrogradskii (1801–1862) in 1836, in the geometrical tracts of Hermann Grassmann (1809–1877) in the early 1840s, and in a brief paper of Arthur Cayley (1821–1895) in 1846. Unfortunately, the first two authors were virtually ignored in their lifetimes. In particular, the work of Grassmann was quite philosophical and extremely difficult to read. Cayley's note merely stated that one can generalize certain results to dimensions greater than three "without recourse to any metaphysical notion with regard to the possibility of a space of four dimensions." Sir William Rowan Hamilton (1805–1865), in an 1841 letter, also noted that "it *must* be possible, in *some* way or other, to introduce not only triplets but *polyplets*, so as in some sense to satisfy the symbolical equation

$$a = (a_1, a_2, \ldots, a_n);$$

a being here one symbol, as indicative of one (complex) thought; and $a_1, a_2, \ldots, a_n$ denoting n real numbers, positive or negative."

Hamilton, whose work on quaternions will be mentioned later, and who spent much of his professional life as the Royal Astronomer of Ireland, is most famous for his work in dynamics. As Erwin Shrödinger wrote, "the Hamiltonian principle has become the cornerstone of modern physics, the thing with which a physicist expects *every* physical phenomenon to be in conformity."

in which the force is acting, and with the length of the arrow representing the magnitude of the force. Such an arrow is a *force vector.*

Using a rectangular coordinate system in the plane, note that if we consider a force vector to start from the origin (0, 0), then the vector is completely determined by the coordinates of the point at the tip of the arrow. Thus we can consider each ordered pair in $\mathbb{R}^2$ to represent a vector in the plane as well as a point in the plane. When we wish to regard an ordered pair as a vector, we will use square brackets, rather than parentheses, to indicate this. Also, we often will write vectors as columns of numbers rather than as rows, and bracket notation is traditional for columns. Thus we speak of the *point* (1, 2) in $\mathbb{R}^2$ and of the *vector* [1, 2] in $\mathbb{R}^2$. To represent the *point* (1, 2) in the plane, we make a dot at the appropriate place, whereas if we wish to represent the *vector* [1, 2], we draw an arrow emanating from the origin with its tip at the place where we would plot the point (1, 2). Mathematically, there is no distinction between (1, 2) and [1, 2]. The different notations merely indicate different views of the same member of $\mathbb{R}^2$. This is illustrated in Figure 1.3. A similar observation holds for 3-space. Generalizing, each n-tuple of real numbers can be viewed both as a point $(x_1, x_2, \ldots, x_n)$ and as a vector $[x_1, x_2, \ldots, x_n]$ in $\mathbb{R}^n$. We use boldface letters such as $\mathbf{a} = [a_1, a_2]$, $\mathbf{v} = [v_1, v_2, v_3]$, and $\mathbf{x} = [x_1, x_2, \ldots, x_n]$ to denote vectors. In written work, it is customary to place an arrow over a letter to denote a vector, as in $\vec{a}$, $\vec{v}$, and $\vec{x}$. The ith entry x_i in such a vector is the ***i*th component** of the vector. Even the real numbers in $\mathbb{R}$ can be regarded both as points and as vectors. When we are not regarding a real number as either a point or a vector, we refer to it as a *scalar.*

Two vectors $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ and $\mathbf{w} = [w_1, w_2, \ldots, w_m]$ are **equal** if $n = m$ and $v_i = w_i$ for each i.

A vector containing only zeros as components is called a **zero vector** and is denoted by $\mathbf{0}$. Thus, in $\mathbb{R}^2$ we have $\mathbf{0} = [0, 0]$ whereas in $\mathbb{R}^4$ we have $\mathbf{0} = [0, 0, 0, 0]$.

When denoting a vector $\mathbf{v}$ in $\mathbb{R}^n$ geometrically by an arrow in a figure, we say that the vector is in *standard position* if it starts at the origin. If we draw an

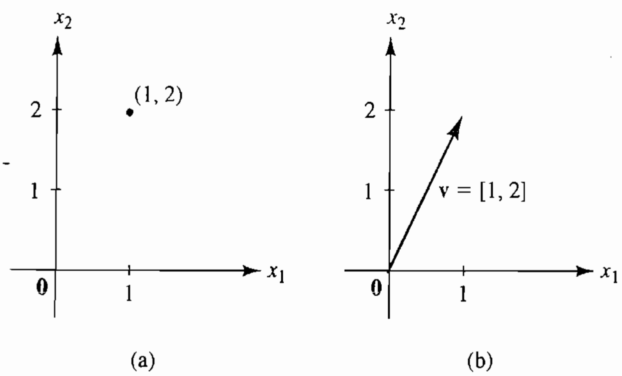

FIGURE 1.3
Two views of the same member of $\mathbb{R}^2$: (a) the point (1, 2); (b) the vector $\mathbf{v} = [1, 2]$.

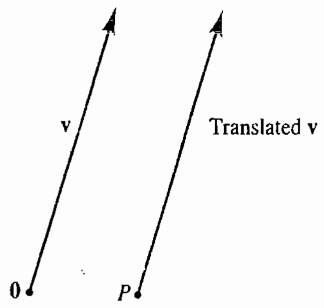

FIGURE 1.4
v translated to P.

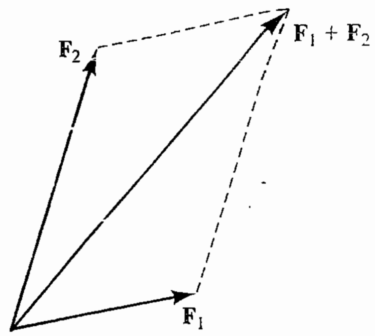

FIGURE 1.5
The vector sum $\mathbf{F}_1 + \mathbf{F}_2$.

arrow having the same length and parallel to the arrow representing **v** but starting at a point P other than the origin, we refer to the arrow as **v** *translated to P*. This is illustrated in Figure 1.4. Note that we did not draw any coordinate axes; we only marked the origin **0** and drew the two arrows. Thus we can consider Figure 1.4 to represent a vector **v** in $\mathbb{R}^2$, $\mathbb{R}^3$, or indeed in $\mathbb{R}^n$ for $n \geq 2$. We will often leave out axes when they are not necessary for our understanding. This makes our figures both less cluttered and more general.

Vector Algebra

Physicists tell us that if two forces corresponding to force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a body at the same time, then the two forces can be replaced by a single force, the *resultant force*, which has the same effect as the original two forces. The force vector for this resultant force is the diagonal of the parallelogram having the force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ as edges, as illustrated in Figure 1.5. It is natural to consider this resultant force vector to be the *sum* $\mathbf{F}_1 + \mathbf{F}_2$ of the two original force vectors, and it is so labeled in Figure 1.5.

HISTORICAL NOTE THE CONCEPT OF A VECTOR in its earliest manifestation comes from physical considerations. In particular, there is evidence of velocity being thought of as a vector—a quantity with magnitude and direction—in Greek times. For example, in the treatise *Mechanica* by an unknown author in the fourth century B.C. is written: "When a body is moved in a certain ratio (i.e., has two linear movements in a constant ratio to one another), the body must move in a straight line, and this straight line is the diagonal of the parallelogram formed from the straight lines which have the given ratio." Heron of Alexandria (first century A.D.) gave a proof of this result when the directions were perpendicular. He showed that if a point A moves with constant velocity over a line AB while at the same time the line AB moves with constant velocity along the parallel lines AC and BD so that it always remains parallel to its original position, and that if the time A takes to reach B is the same as the time AB takes to reach CD, then in fact the point A moves along the diagonal AD.

This basic idea of adding two motions vectorially was generalized from velocities to physical forces in the sixteenth and seventeenth centuries. One example of this practice is found as Corollary 1 to the Laws of Motion in Isaac Newton's *Principia*, where he shows that "a body acted on by two forces simultaneously will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately."

We can visualize two vectors with different directions and emanating from a point P in Euclidean 2-space or 3-space as determining a plane. It is pedagogically useful to do this for n-space for any $n \geq 2$ and show helpful figures on our pages. Motivated by our discussion of force vectors above, we consider the *sum* of two vectors $\mathbf{v}$ and $\mathbf{w}$ starting at a point P to be the vector starting at P that forms the diagonal of the parallelogram with a vertex at P and having edges represented by $\mathbf{v}$ and $\mathbf{w}$, as illustrated in Figure 1.6, where we take the vectors in $\mathbb{R}^n$ in standard position starting at $\mathbf{0}$. Thus we have a geometric understanding of *vector addition* in $\mathbb{R}^n$. We have labeled as *translated* $\mathbf{v}$ and *translated* $\mathbf{w}$ the sides of the parallelogram opposite the vectors $\mathbf{v}$ and $\mathbf{w}$.

Note that arrows along opposite sides of the parallelogram point in the same direction and have the same length. Thus, as a force vector, the translation of $\mathbf{v}$ is considered to be equivalent to the vector $\mathbf{v}$, and the same is true for $\mathbf{w}$ and its translation. We can think of obtaining the vector $\mathbf{v} + \mathbf{w}$ by drawing the arrow $\mathbf{v}$ from $\mathbf{0}$ and then drawing the arrow $\mathbf{w}$ translated to start from the tip of $\mathbf{v}$ as shown in Figure 1.6. The vector from $\mathbf{0}$ to the tip of the translated $\mathbf{w}$ is then $\mathbf{v} + \mathbf{w}$. This is often a useful way to regard $\mathbf{v} + \mathbf{w}$. To add three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ geometrically, we translate $\mathbf{v}$ to start at the tip of $\mathbf{u}$ and then translate $\mathbf{w}$ to start at the tip of the translated $\mathbf{v}$. The sum $\mathbf{u} + \mathbf{v} + \mathbf{w}$ then begins at the origin where $\mathbf{u}$ starts, and ends at the tip of the translated $\mathbf{w}$, as indicated in Figure 1.7.

The difference $\mathbf{v} - \mathbf{w}$ of two vectors in $\mathbb{R}^n$ is represented geometrically by the arrow from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$, as shown in Figure 1.8. Here $\mathbf{v} - \mathbf{w}$ is the vector that, when added to $\mathbf{w}$, yields $\mathbf{v}$. The dashed arrow in Figure 1.8 shows $\mathbf{v} - \mathbf{w}$ in standard position.

If we are pushing a body with a force vector $\mathbf{F}$ and we wish to "double the force"—that is, we want to push in the same direction but twice as hard—then it is natural to denote the doubled force vector by $2\mathbf{F}$. If instead we want to push the body in the opposite direction with one-third the force, we denote the new force vector by $-\frac{1}{3}\mathbf{F}$. Generalizing, we consider the product $r\mathbf{v}$ of a scalar r times a vector $\mathbf{v}$ in $\mathbb{R}^n$ to be represented by the arrow whose length is $|r|$ times the length of $\mathbf{v}$ and which has the same direction as $\mathbf{v}$ if $r > 0$ but the opposite direction if $r < 0$. (See Figure 1.9 for an illustration.) Thus we have a geometric interpretation of *scalar multiplication* in $\mathbb{R}^n$—that is, of multiplication of a vector in $\mathbb{R}^n$ by a scalar.

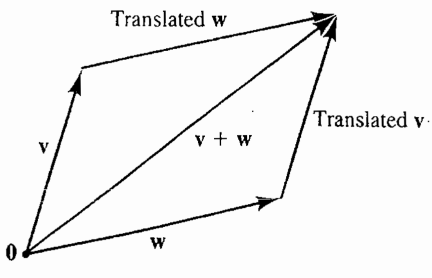

FIGURE 1.6
Representation of $\mathbf{v} + \mathbf{w}$ in $\mathbb{R}^n$.

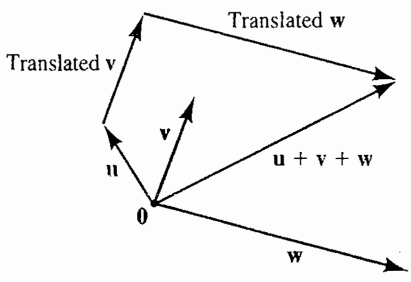

FIGURE 1.7
Representation of $\mathbf{u} + \mathbf{v} + \mathbf{w}$ in $\mathbb{R}^n$.

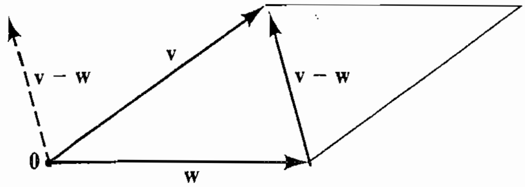

FIGURE 1.8
The vector v − w.

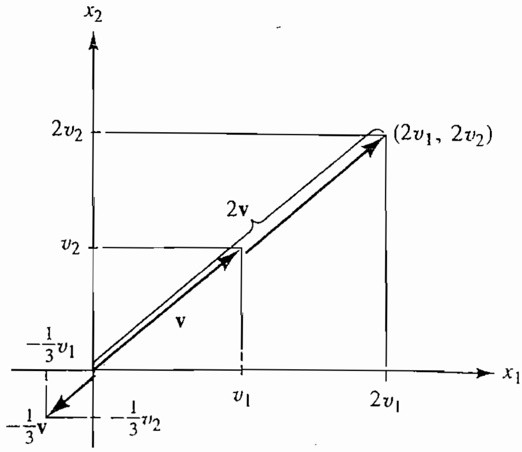

FIGURE 1.9
Computation of $r\mathbf{v}$ in $\mathbb{R}^2$.

Taking a vector $\mathbf{v} = [v_1, v_2]$ in $\mathbb{R}^2$ and any scalar r, we would like to be able to compute $r\mathbf{v}$ algebraically as an element (ordered pair) in $\mathbb{R}^2$, and not just represent it geometrically by an arrow. Figure 1.9 shows the vector $2\mathbf{v}$ which points in the same direction as $\mathbf{v}$ but is twice as long, and shows that we have $2\mathbf{v} = [2v_1, 2v_2]$. It also indicates that if we multiply all components of $\mathbf{v}$ by $-\frac{1}{3}$, the resulting vector has direction opposite to the direction of $\mathbf{v}$ and length equal to $\frac{1}{3}$ the length of $\mathbf{v}$. Similarly, if we take two vectors $\mathbf{v} = [v_1, v_2]$ and $\mathbf{w} = [w_1, w_2]$ in $\mathbb{R}^2$, we would like to be able to compute $\mathbf{v} + \mathbf{w}$ algebraically as an element (ordered pair) in $\mathbb{R}^2$. Figure 1.10 indicates that we have $\mathbf{v} + \mathbf{w} = [v_1 + w_1, v_2 + w_2]$—that is, we can simply add corresponding components. With these figures to guide us, we formally define some algebraic operations with vectors in $\mathbb{R}^n$.

DEFINITION 1.1 Vector Algebra in $\mathbb{R}^n$

Let $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ and $\mathbf{w} = [w_1, w_2, \ldots, w_n]$ be vectors in $\mathbb{R}^n$. The vectors are added and subtracted as follows:

Vector addition: $\mathbf{v} + \mathbf{w} = [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]$

Vector subtraction: $\mathbf{v} - \mathbf{w} = [v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n]$

If r is any scalar, the vector $\mathbf{v}$ is multiplied by r as follows:

Scalar multiplication: $r\mathbf{v} = [rv_1, rv_2, \ldots, rv_n]$

As a natural extension of Definition 1.1, we can combine three or more vectors in $\mathbb{R}^n$ using addition or subtraction by simply adding or subtracting their corresponding components. When a scalar in such a combination is negative, as in $4\mathbf{u} + (-7)\mathbf{v} + 2\mathbf{w}$, we usually abbreviate by subtraction, writing $4\mathbf{u} - 7\mathbf{v} + 2\mathbf{w}$. We write $-\mathbf{v}$ for $(-1)\mathbf{v}$.

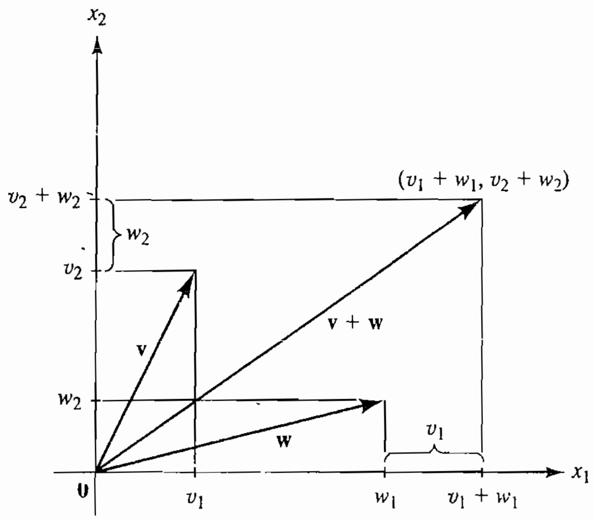

FIGURE 1.10
Computation of v + w in $\mathbb{R}^2$.

EXAMPLE 1 Let $\mathbf{v} = [-3, 5, -1]$ and $\mathbf{w} = [4, 10, -7]$ in $\mathbb{R}^3$. Compute $5\mathbf{v} - 3\mathbf{w}$.

SOLUTION We compute

$$\begin{aligned} 5\mathbf{v} - 3\mathbf{w} &= 5[-3, 5, -1] - 3[4, 10, -7] \\ &= [-15, 25, -5] - [12, 30, -21] \\ &= [-27, -5, 16]. \end{aligned}$$ ∎

EXAMPLE 2 For vectors **v** and **w** in $\mathbb{R}^n$ pointing in different directions from the origin, represent geometrically $5\mathbf{v} - 3\mathbf{w}$.

SOLUTION This is done in Figure 1.11. ∎

The analogues of many familiar algebraic laws for addition and multiplication of scalars also hold for vector addition and scalar multiplication. For convenience, we gather them in a theorem.

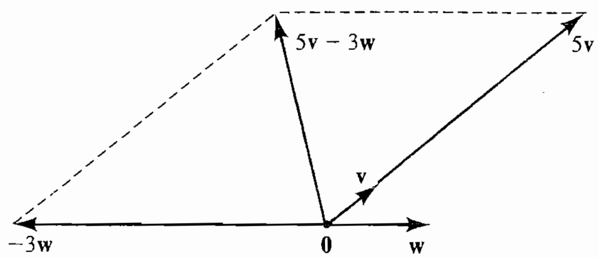

FIGURE 1.11
5v − 3w in $\mathbb{R}^n$.

THEOREM 1.1 Properties of Vector Algebra in $\mathbb{R}^n$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be any vectors in $\mathbb{R}^n$, and let r and s be any scalars in $\mathbb{R}$.

Properties of Vector Addition

A1 $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	**An associative law**
A2 $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$	**A commutative law**
A3 $\mathbf{0} + \mathbf{v} = \mathbf{v}$	**0 as additive identity**
A4 $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$	**$-\mathbf{v}$ as additive inverse of v**

Properties Involving Scalar Multiplication

S1 $r(\mathbf{v} + \mathbf{w}) = r\mathbf{v} + r\mathbf{w}$	**A distributive law**
S2 $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$	**A distributive law**
S3 $r(s\mathbf{v}) = (rs)\mathbf{v}$	**An associative law**
S4 $1\mathbf{v} = \mathbf{v}$	**Preservation of scale**

The eight properties given in Theorem 1.1 are quite easy to prove, and we leave most of them as exercises. The proofs in Examples 3 and 4 are typical.

EXAMPLE 3 Prove property A2 of Theorem 1.1.

SOLUTION Writing

$$\mathbf{v} = [v_1, v_2, \ldots, v_n] \quad \text{and} \quad \mathbf{w} = [w_1, w_2, \ldots, w_n],$$

we have

$$\mathbf{v} + \mathbf{w} = [v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n]$$

and

$$\mathbf{w} + \mathbf{v} = [w_1 + v_1, w_2 + v_2, \ldots, w_n + v_n].$$

These two vectors are equal because $v_i + w_i = w_i + v_i$ for each i. Thus, the commutative law of vector addition follows directly from the commutative law of addition of numbers. ▪

EXAMPLE 4 Prove property S2 of Theorem 1.1.

SOLUTION Writing $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, we have

$$\begin{aligned}(r + s)\mathbf{v} &= (r + s)[v_1, v_2, \ldots, v_n] \\ &= [(r + s)v_1, (r + s)v_2, \ldots, (r + s)v_n] \\ &= [rv_1 + sv_1, rv_2 + sv_2, \ldots, rv_n + sv_n] \\ &= [rv_1, rv_2, \ldots, rv_n] + [sv_1, sv_2, \ldots, sv_n] \\ &= r\mathbf{v} + s\mathbf{v}.\end{aligned}$$

Thus the property $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$ involving vectors follows from the analogous property $(r + s)a_i = ra_i + sa_i$ for numbers. ▪

Parallel Vectors

The geometric significance of multiplication of a vector by a scalar, as illustrated in Figure 1.9, leads us to this characterization of parallel vectors.

DEFINITION 1.2 Parallel Vectors

Two nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ are parallel, and we write $\mathbf{v} \parallel \mathbf{w}$, if one is a scalar multiple of the other. If $\mathbf{v} = r\mathbf{w}$ with $r > 0$, then $\mathbf{v}$ and $\mathbf{w}$ have the same direction; if $r < 0$, then $\mathbf{v}$ and $\mathbf{w}$ have opposite directions.

EXAMPLE 5 Determine whether the vectors $\mathbf{v} = [2, 1, 3, -4]$ and $\mathbf{w} = [6, 3, 9, -12]$ are parallel.

SOLUTION We put $\mathbf{v} = r\mathbf{w}$ and try to solve for r. This gives rise to four component equations:

$$2 = 6r, \qquad 1 = 3r, \qquad 3 = 9r, \qquad -4 = -12r.$$

Because $r = \frac{1}{3} > 0$ is a common solution to the four equations, we conclude that $\mathbf{v}$ and $\mathbf{w}$ are parallel and have the same direction. ▪

Linear Combinations of Vectors

Definition 1.1 describes how to add or subtract two vectors, but as we remarked following the definition, we can use these operations to combine three or more vectors also. We give a formal extension of that definition.

DEFINITION 1.3 Linear Combination

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ and scalars $r_1, r_2, \ldots, r_k$ in $\mathbb{R}$, the vector

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$$

is a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ with **scalar coefficients** $r_1, r_2, \ldots, r_k$.

The vectors $[1, 0]$ and $[0, 1]$ play a very important role in $\mathbb{R}^2$. Every vector $\mathbf{b}$ in $\mathbb{R}^2$ can be expressed as a linear combination of these two vectors in a *unique* way—namely, $\mathbf{b} = [b_1, b_2] = r_1[1, 0] + r_2[0, 1]$ if and only if $r_1 = b_1$ and $r_2 = b_2$. We call $[1, 0]$ and $[0, 1]$ the **standard basis vectors in** $\mathbb{R}^2$. They are often denoted by $\mathbf{i} = [1, 0]$ and $\mathbf{j} = [0, 1]$, as shown in Figure 1.12(a). Thus in $\mathbb{R}^2$, we may write the vector $[b_1, b_2]$ as $b_1\mathbf{i} + b_2\mathbf{j}$. Similarly, we have three **standard basis vectors in** $\mathbb{R}^3$—namely,

$$\mathbf{i} = [1, 0, 0], \quad \mathbf{j} = [0, 1, 0], \quad \text{and} \quad \mathbf{k} = [0, 0, 1],$$

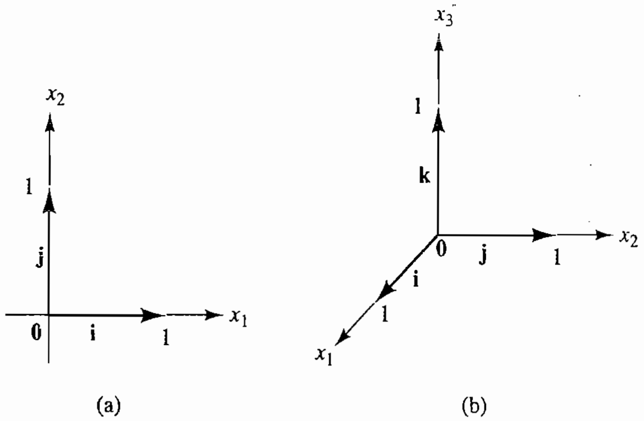

FIGURE 1.12
(a) Standard basis vectors in $\mathbb{R}^2$; (b) standard basis vectors in $\mathbb{R}^3$.

as shown in Figure 1.12(b). Every vector in $\mathbb{R}^3$ can be expressed uniquely as a linear combination of **i**, **j**, and **k**. For example, we have $[3, -2, 6] = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$. For $n > 3$, we denote the **rth standard basis vector**, having 1 as the rth component and zeros elsewhere, by

$$\mathbf{e}_r = [0, 0, \ldots, 0, \underset{\substack{\uparrow \\ r\text{th component}}}{1}, 0, \ldots, 0].$$

We then have

$$\mathbf{b} = [b_1, b_2, \ldots, b_n] = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + \cdots + b_n\mathbf{e}_n.$$

We see that every vector in $\mathbb{R}^n$ appears as a *unique* linear combination of the standard basis vector in $\mathbb{R}^n$.

The Span of Vectors

Let **v** be a vector in $\mathbb{R}^n$. All possible linear combinations of this single vector **v** are simply all possible scalar multiples $r\mathbf{v}$ for all scalars r. If $\mathbf{v} \neq \mathbf{0}$, all scalar multiples of **v** fill a line which we shall call the **line along v**. Figure 1.13(a) shows the line along the vector $[-1, 2]$ in $\mathbb{R}^2$ while Figure 1.13(b) indicates the line along a nonzero vector **v** in $\mathbb{R}^n$.

Note that the line along **v** always contains the origin (the zero vector) because one scalar multiple of **v** is $0\mathbf{v} = \mathbf{0}$.

Now let **v** and **w** be two nonzero and nonparallel vectors in $\mathbb{R}^n$. All possible linear combinations of **v** and **w** are all vectors of the form $r\mathbf{v} + s\mathbf{w}$ for all scalars r and s. As indicated in Figure 1.14, all these linear combinations fill a plane which we call the **plane spanned by v and w**.

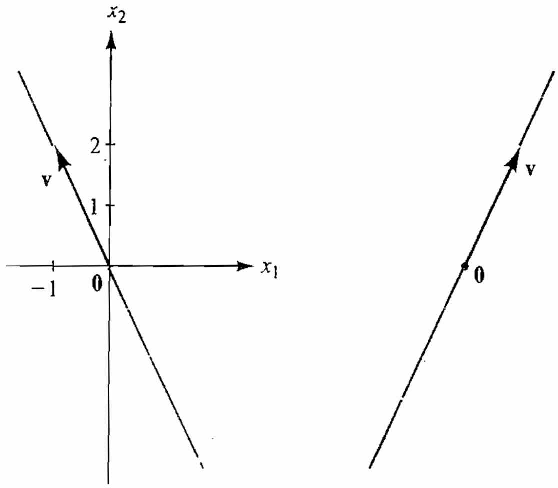

FIGURE 1.13
(a) The line along v in $\mathbb{R}^2$; (b) The line along v in $\mathbb{R}^n$.

EXAMPLE 6 Referring to Figure 1.15(a), estimate scalars r and s such that $r\mathbf{v} + s\mathbf{w} = \mathbf{b}$ for the vectors $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{b}$ all lying in the plane of the paper.

SOLUTION We draw the line along $\mathbf{v}$, the line along $\mathbf{w}$, and parallels to these lines through the tip of the vector $\mathbf{b}$, as shown in Figure 1.15(b). From Figure 1.15(b), we estimate that $\mathbf{b} = 1.5\mathbf{v} - 2.5\mathbf{w}$. ▪

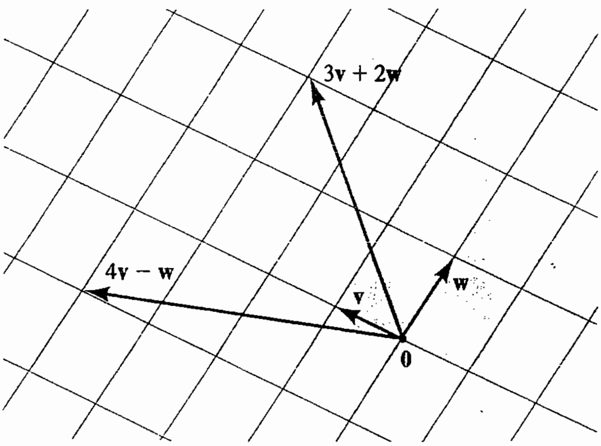

FIGURE 1.14
The plane spanned by v and w.

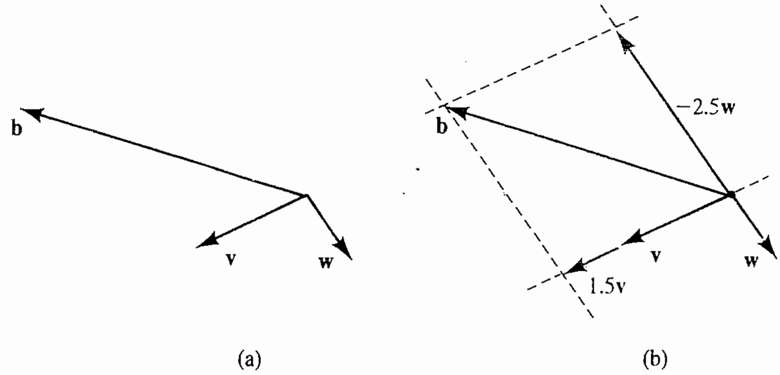

FIGURE 1.15
(a) Vectors v, w, and b; (b) finding r and s so that $\mathbf{b} = r\mathbf{v} + s\mathbf{w}$.

We now give an analytic analogue of Example 6 for two vectors in $\mathbb{R}^2$.

EXAMPLE 7 Let $\mathbf{v} = [1, 3]$ and $\mathbf{w} = [-2, 5]$ in $\mathbb{R}^2$. Find scalars r and s such that $r\mathbf{v} + s\mathbf{w} = [-1, 19]$.

SOLUTION Because $r\mathbf{v} + s\mathbf{w} = r[1, 3] + s[-2, 5] = [r - 2s, 3r + 5s]$, we see that $r\mathbf{v} + s\mathbf{w} = [-1, 19]$ if and only if both equations

$$\begin{aligned} r - 2s &= -1 \\ 3r + 5s &= 19 \end{aligned}$$

are satisfied. Multiplying the first equation by -3 and adding the result to the second equation, we obtain

$$0 + 11s = 22,$$

so $s = 2$. Substituting in the equation $r - 2s = -1$, we find that $r = 3$. ∎

We note that the components -1 and 19 of the vector $[-1, 19]$ appear on the right-hand side of the system of two linear equations in Example 7. If we replace -1 by b_1 and 19 by b_2, the same operations on the equations will enable us to solve for the scalars r and s in terms of b_1 and b_2 (see Exercise 42). This shows that all linear combinations of $\mathbf{v}$ and $\mathbf{w}$ do indeed fill the plane $\mathbb{R}^2$.

Example 7 indicates that an attempt to express a vector $\mathbf{b}$ as a linear combination of given vectors corresponds to an attempt to find a solution of a system of linear equations. This parallel is even more striking if we write our vectors as *columns* of numbers rather than as ordered *rows* of numbers—that is, as **column vectors** rather than as **row vectors**. For example, if we write the vectors $\mathbf{v}$ and $\mathbf{w}$ in Example 7 as columns so that

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} -2 \\ 5 \end{bmatrix},$$

and also rewrite $[-1, 19]$ as a column vector, then the row-vector equation $r\mathbf{v} + s\mathbf{w} = [-1, 19]$ in the statement of Example 7 becomes

$$r\begin{bmatrix} 1 \\ 3 \end{bmatrix} + s\begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 19 \end{bmatrix}.$$

Notice that the numbers in this column-vector equation are in the same positions relative to each other as they are in the *system of linear equations*

$$\begin{aligned} r - 2s &= -1 \\ 3r + 5s &= 19 \end{aligned}$$

that we solved in Example 7. Every system of linear equations can be rewritten in this fashion as a single column-vector equation. Exercises 35–38 provide practice in this. Finding scalars $r_1, r_2, \ldots, r_k$ such that $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{b}$ for given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and $\mathbf{b}$ in $\mathbb{R}^n$ is a fundamental computation in linear algebra. Section 1.4 describes an algorithm for finding all possible such scalars $r_1, r_2, \ldots, r_k$.

The preceding paragraph indicates that often it will be natural for us to think of vectors in $\mathbb{R}^n$ as column vectors rather than as row vectors.

The **transpose** of a row vector $\mathbf{v}$ is defined to be the corresponding column vector, and is denoted by $\mathbf{v}^T$. Similarly, the transpose of a column vector is the corresponding row vector. For example,

$$[-1, 4, 15, -7]^T = \begin{bmatrix} -1 \\ 4 \\ 15 \\ -7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -30 \\ 45 \end{bmatrix}^T = [2, -30, 45].$$

Note that for all vectors $\mathbf{v}$ we have $(\mathbf{v}^T)^T = \mathbf{v}$. As illustrated following Example 7, column vectors are often useful. In fact, some authors always regard every vector $\mathbf{v}$ in $\mathbb{R}^n$ as a column vector. Because it takes so much page space to write column vectors, these authors may describe $\mathbf{v}$ by giving the row vector $\mathbf{v}^T$. We do not follow this practice; we will write vectors in $\mathbb{R}^n$ as either row or column vectors depending on the context.

Continuing our geometric discussion, we expect that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three nonzero vectors in $\mathbb{R}^n$ such that $\mathbf{u}$ and $\mathbf{v}$ are not parallel and also $\mathbf{w}$ is not a vector in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$, then the set of all linear combinations of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ will fill a three-dimensional portion of $\mathbb{R}^n$—that is, a portion of $\mathbb{R}^n$ that looks just like $\mathbb{R}^3$. We consider the set of these linear combinations to be *spanned* by $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. We make the following definition.

DEFINITION 1.4 Span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. The **span** of these vectors is the set of all linear combinations of them and is denoted by $\text{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$. In set notation,

$$\text{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k) = \{r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k \mid r_1, r_2, \ldots, r_k \in \mathbb{R}\}.$$

It is important to note that $\text{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ in $\mathbb{R}^n$ may not fill what we intuitively consider to be a k-dimensional portion of $\mathbb{R}^n$. For example, in $\mathbb{R}^2$ we see that $\text{sp}([1, -2], [-3, 6])$ is just the one-dimensional line along $[1, -2]$ because $[-3, 6] = -3[1, -2]$ already lies in $\text{sp}([1, -2])$. Similarly, if $\mathbf{v}_3$ is a vector in $\text{sp}(\mathbf{v}_1, \mathbf{v}_2)$, then $\text{sp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \text{sp}(\mathbf{v}_1, \mathbf{v}_2)$ and so $\text{sp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is not three-dimensional. Section 2.1 will deal with this kind of *dependency* among vectors. As a result of our work there, we will be able to define dimensionality.

SUMMARY

1. *Euclidean n-space* $\mathbb{R}^n$ consists of all ordered n-tuples of real numbers. Each n-tuple $\mathbf{x}$ can be regarded as a *point* $(x_1, x_2, \ldots, x_n)$ and represented graphically as a dot, or regarded as a vector $[x_1, x_2, \ldots, x_n]$ and represented by an arrow. The n-tuple $\mathbf{0} = [0, 0, \ldots, 0]$ is the *zero vector*. A real number $r \in \mathbb{R}$ is called a *scalar*.
2. Vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ can be added and subtracted, and each can be multiplied by a scalar $r \in \mathbb{R}$. In each case, the operation is performed on the components, and the resulting vector is again in $\mathbb{R}^n$. Properties of these operations are summarized in Theorem 1.1. Graphic interpretations are shown in Figures 1.6, 1.8, and 1.9.
3. Two nonzero vectors in $\mathbb{R}^n$ are *parallel* if one is a scalar multiple of the other.
4. A *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ is a vector of the form $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$, where each r_i is a scalar. The set of all such linear combinations is the *span* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\text{sp}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$.
5. Every vector in $\mathbb{R}^n$ can be expressed uniquely as a linear combination of the *standard basis vectors* $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, where $\mathbf{e}_i$ has 1 as its ith component and zeros for all other components.

EXERCISES

In Exercises 1–4, compute $\mathbf{v} + \mathbf{w}$ *and* $\mathbf{v} - \mathbf{w}$ *for the given vectors* $\mathbf{v}$ *and* $\mathbf{w}$. *Then draw coordinate axes and sketch, using your answers, the vectors* $\mathbf{v}$, $\mathbf{w}$, $\mathbf{v} + \mathbf{w}$, *and* $\mathbf{v} - \mathbf{w}$.

1. $\mathbf{v} = [2, -1]$, $\mathbf{w} = [-3, -2]$
2. $\mathbf{v} = [1, 3]$, $\mathbf{w} = [-2, 5]$
3. $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$
4. $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, $\mathbf{w} = 3\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$

In Exercises 5–8, let $\mathbf{u} = [-1, 3, -2]$, $\mathbf{v} = [4, 0, -1]$, *and* $\mathbf{w} = [-3, -1, 2]$. *Compute the indicated vector.*

5. $3\mathbf{u} - 2\mathbf{v}$
6. $\mathbf{u} + 2(\mathbf{v} - 4\mathbf{w})$
7. $\mathbf{u} + \mathbf{v} - \mathbf{w}$
8. $4(3\mathbf{u} + 2\mathbf{v} - 5\mathbf{w})$