1.2 THE NORM AND THE DOT PRODUCT

The Magnitude of a Vector

The *magnitude* $\|\mathbf{v}\|$ of $\mathbf{v} = [v_1, v_2]$ is considered to be the length of the arrow in Figure 1.18. Using the Pythagorean theorem, we have

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}. \tag{1}$$

EXAMPLE 1 Represent the vector $\mathbf{v} = [3, -4]$ geometrically, and find its magnitude.

SOLUTION The vector $[3, -4]$ has magnitude

$$\|\mathbf{v}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$$

and is shown in Figure 1.19. ∎

In Figure 1.20, the *magnitude* $\|\mathbf{v}\|$ of a vector $\mathbf{v} = [v_1, v_2, v_3]$ in $\mathbb{R}^3$ appears as the length of the hypotenuse of a right triangle whose altitude is $|v_3|$ and whose base in the x_1,x_2-plane has length $\sqrt{v_1^2 + v_2^2}$. Using the Pythagorean theorem, we obtain

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{2}$$

EXAMPLE 2 Represent the vector $\mathbf{v} = [2, 3, 4]$ geometrically, and find its magnitude.

SOLUTION The vector $\mathbf{v} = [2, 3, 4]$ has magnitude $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$ and is represented in Figure 1.21. ∎

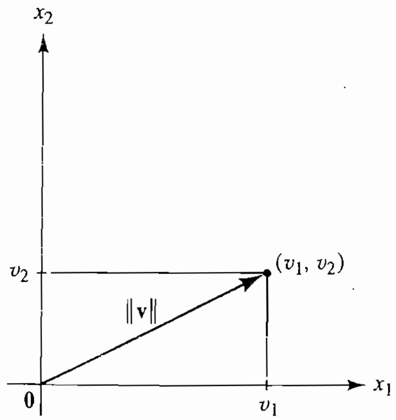

FIGURE 1.18
The magnitude of v in $\mathbb{R}^2$.

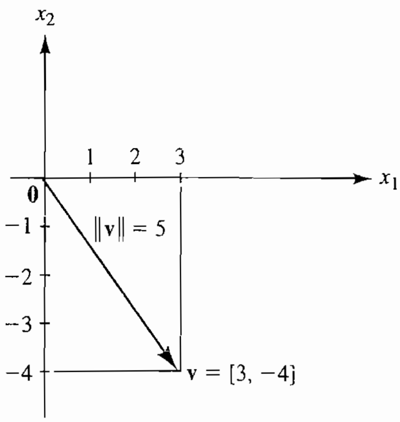

FIGURE 1.19
The magnitude of [3, −4].

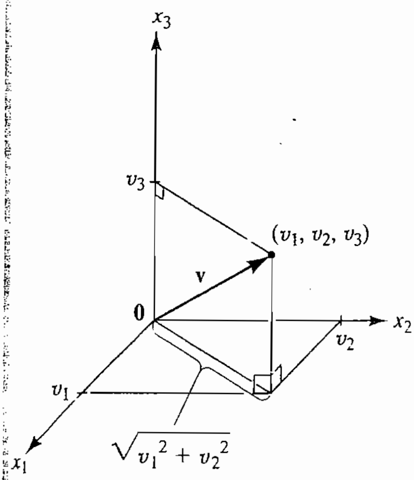

FIGURE 1.20
The magnitude of v in $\mathbb{R}^3$.

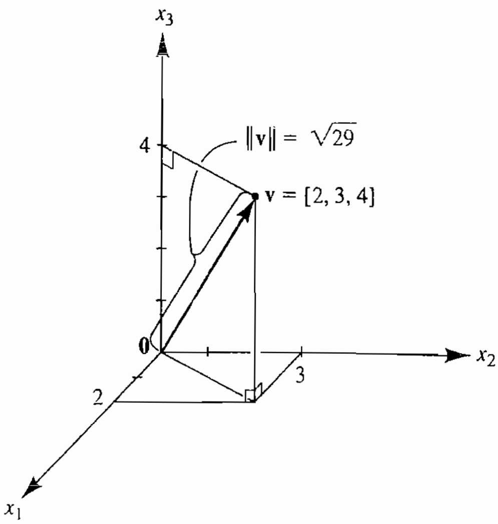

FIGURE 1.21
The magnitude of [2, 3, 4].

The magnitude of a vector is also called the *norm* or the *length* of the vector. As suggested by Eqs. (1) and (2), we define the norm $\|\mathbf{v}\|$ of a vector **v** in $\mathbb{R}^n$ as follows.

DEFINITION 1.5 Norm or Magnitude of a Vector in $\mathbb{R}^n$

Let $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ be a vector in $\mathbb{R}^n$. The **norm** or **magnitude** of **v** is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}. \quad (3)$$

EXAMPLE 3 Find the magnitude of the vector $\mathbf{v} = [-2, 1, 3, -1, 4, 2, 1]$.

SOLUTION We have

$$\|\mathbf{v}\| = \sqrt{(-2)^2 + 1^2 + 3^2 + (-1)^2 + 4^2 + 2^2 + 1^2} = \sqrt{36} = 6.$$ ∎

Here are some properties of this norm operation.

THEOREM 1.2 Properties of the Norm in $\mathbb{R}^n$

For all vectors **v** and **w** in $\mathbb{R}^n$ and for all scalars r, we have

1. $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$ **Positivity**
2. $\|r\mathbf{v}\| = |r|\,\|\mathbf{v}\|$ **Homogeneity**
3. $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ **Triangle inequality**

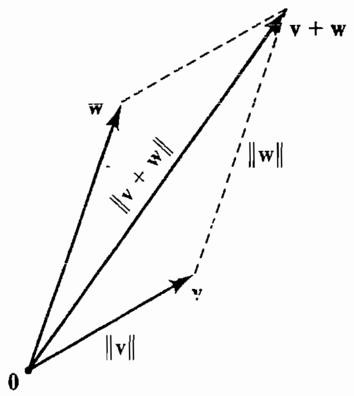

FIGURE 1.22
The triangle inequality.

Proofs of Properties 1 and 2 follow immediately from Definition 1.5 and appear as exercises at the end of this section. Figure 1.22 shows why Property 3 is called the triangle inequality; geometrically, it states that the length of a side of a triangle is less than or equal to the sum of the lengths of the other two sides. Although this seems obvious to us from Figure 1.22, we really should prove it—at least for $n > 3$, where we simply extended our definition of $\|\mathbf{v}\|$ for $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$ without any further geometric justification. A proof of the triangle inequality is given at the close of this section.

Unit Vectors

A vector in $\mathbb{R}^n$ is a **unit vector** if it has magnitude 1. Given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, a unit vector having the same direction as $\mathbf{v}$ is given by $(1/\|\mathbf{v}\|)\mathbf{v}$.

EXAMPLE 4 Find a unit vector having the same direction as $\mathbf{v} = [2, 1, -3]$, and find a vector of magnitude 3 having direction opposite to $\mathbf{v}$.

SOLUTION Because $\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-3)^2} = \sqrt{14}$, we see that $\mathbf{u} = (1/\sqrt{14})[2, 1, -3]$ is the unit vector with the same direction as $\mathbf{v}$, and $-3\mathbf{u} = (-3/\sqrt{14})[2, 1, -3]$ is the other required vector. ■

The two-component unit vectors are precisely the vectors that extend from the origin to the unit circle $x^2 + y^2 = 1$ with center (0, 0) and radius 1 in $\mathbb{R}^2$. (See Figure 1.23a.) The three-component unit vectors extend from (0, 0, 0) to the unit sphere in $\mathbb{R}^3$, as illustrated in Figure 1.23(b).

Note that the standard basis vectors $\mathbf{i}$ and $\mathbf{j}$ in $\mathbb{R}^2$, as well as $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ in $\mathbb{R}^3$, are unit vectors. In fact, the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for $\mathbb{R}^n$ are unit vectors, because each has zeros in all components except for one component of 1. For this reason, these standard basis vectors are also called **unit coordinate vectors**.

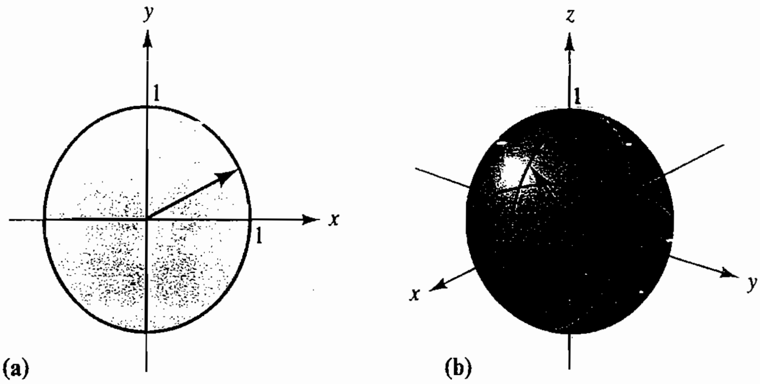

FIGURE 1.23
(a) Typical unit vector in $\mathbb{R}^2$; (b) typical unit vector in $\mathbb{R}^3$.

The Dot Product

The dot product of two vectors is a scalar that we will encounter as we now try to define the angle θ between two vectors $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ and $\mathbf{w} = [w_1, w_2, \ldots, w_n]$ in $\mathbb{R}^n$, shown symbolically in Figure 1.24. To motivate the definition of θ, we will use the law of cosines for the triangle symbolized in Figure 1.24. Using our definition of the norm of a vector in $\mathbb{R}^n$ to compute the lengths of the sides of the triangle, the law of cosines yields

$$\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \|\mathbf{v} - \mathbf{w}\|^2 + 2\|\mathbf{v}\|\,\|\mathbf{w}\|(\cos\theta)$$

or

$$\begin{aligned} &v_1^2 + \cdots + v_n^2 + w_1^2 + \cdots + w_n^2 \\ &\qquad = (v_1 - w_1)^2 + \cdots + (v_n - w_n)^2 + 2\|\mathbf{v}\|\,\|\mathbf{w}\|(\cos\theta). \end{aligned} \tag{4}$$

After computing the squares on the right-hand side of Eq. (4) and simplifying, we obtain

$$\|\mathbf{v}\|\,\|\mathbf{w}\|(\cos\theta) = v_1w_1 + \cdots + v_nw_n. \tag{5}$$

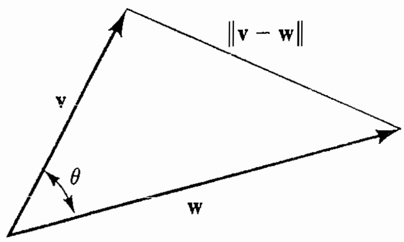

FIGURE 1.24
The angle between v and w.

The sum of products of corresponding components in the vectors **v** and **w** on the right-hand side of Eq. (5) is frequently encountered, and is given a special name and notation.

DEFINITION 1.6 The Dot Product

The *dot product* of vectors $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ and $\mathbf{w} = [w_1, w_2, \ldots, w_n]$ in $\mathbb{R}^n$ is the scalar given by

$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n. \tag{6}$$

The dot product is sometimes called the *inner product* or the *scalar product.* To avoid possible confusion with scalar multiplication, we shall never use the latter term.

In view of Definition 1.6, we can write Eq. (5) as

$$\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|\,(\cos\theta). \tag{7}$$

Equation (7) suggests the following definition of the angle θ between two vectors **v** and **w** in $\mathbb{R}^n$.

The **angle** between nonzero vectors **v** and **w** is $\arccos\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right)$. (8)

Expression (8) makes sense, provided that

$$-1 \le \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|} \le 1, \tag{9}$$

so that we can indeed compute the arccosine of $(\mathbf{v}\cdot\mathbf{w})/(\|\mathbf{v}\|\,\|\mathbf{w}\|)$. This inequality (9) is usually rewritten in the form

$$|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\|. \quad \textbf{Schwarz inequality} \tag{10}$$

We obtained it by assuming that Figure 1.24 is an appropriate representation for vectors **v** and **w** in $\mathbb{R}^n$. We give a purely algebraic proof of it at the end of this section to validate the definition in expression (8).

EXAMPLE 5 Find the angle θ between the vectors [1, 2, 0, 2] and [−3, 1, 1, 5] in $\mathbb{R}^4$.

SOLUTION We have

$$\cos\theta = \frac{[1, 2, 0, 2]\cdot[-3, 1, 1, 5]}{\sqrt{1^2+2^2+0^2+2^2}\sqrt{(-3)^2+1^2+1^2+5^2}} = \frac{9}{(3)(6)} = \frac{1}{2}.$$

Thus, $\theta = 60°$. ▪

Equation 7 gives a geometric meaning for the dot product.

> The dot product of two vectors is equal to the product of their magnitudes with the cosine of the angle between them.

THEOREM 1.3 Properties of the Dot Product in $\mathbb{R}^n$

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$ and let r be any scalar in $\mathbb{R}$. The following properties hold:

D1 $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$, **Commutative law**

D2 $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$, **Distributive law**

D3 $r(\mathbf{v} \cdot \mathbf{w}) = (r\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (r\mathbf{w})$, **Homogeneity**

D4 $\mathbf{v} \cdot \mathbf{v} \geq 0$, and $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$. **Positivity**

Verification of all of the properties in Theorem 1.3 is straightforward, as illustrated in the following example.

HISTORICAL NOTE THE SCHWARZ INEQUALITY is due independently to Augustin-Louis Cauchy (1789–1857) (see note on page 3), Hermann Amandus Schwarz (1843–1921), and Viktor Yakovlevich Bunyakovsky (1804–1889).

It was first stated as a theorem about coordinates in an appendix to Cauchy's 1821 text for his course on analysis at the Ecole Polytechnique, as follows:

$$\left|a\alpha + a'\alpha' + a''\alpha'' + \cdots\right| \leq \sqrt{a^2 + a'^2 + a''^2 + \cdots}\,\sqrt{\alpha^2 + \alpha'^2 + \alpha''^2 + \cdots}\,.$$

Cauchy's proof follows from the algebraic identity

$$(a\alpha + a'\alpha' + a''\alpha'' + \cdots)^2 + (a\alpha' - a'\alpha)^2 + (a\alpha'' - a''\alpha)^2 + \cdots + (a'\alpha'' - a''\alpha')^2 + \cdots$$
$$= (a^2 + a'^2 + a''^2 + \cdots)(\alpha^2 + \alpha'^2 + \alpha''^2 + \cdots).$$

Bunyakovsky proved the inequality for functions in 1859; that is, he stated the result

$$\left[\int_a^b f(x)g(x)\,dx\right]^2 \leq \int_a^b f^2(x)\,dx \cdot \int_a^b g^2(x)\,dx,$$

where we can consider $\int_a^b f(x)g(x)\,dx$ to be the inner product of the functions $f(x)$, $g(x)$ in the vector space of continuous functions on $[a, b]$. Bunyakovsky served as vice-president of the St. Petersburg Academy of Sciences from 1864 until his death. In 1875, the Academy established a mathematics prize in his name in recognition of his 50 years of teaching and research.

Schwarz stated the inequality in 1884. In his case, the vectors were functions ϕ, X of two variables in a region T in the plane, and the inner product of these functions was given by $\iint_T \phi X\,dx\,dy$, where this integral is assumed to exist. The inequality then states that

$$\left|\iint_T \phi X\,dx\,dy\right| \leq \sqrt{\iint_T \phi^2\,dx\,dy} \cdot \sqrt{\iint_T X^2\,dx\,dy}.$$

Schwarz's proof is similar to the one given in the text (page 29). Schwarz was the leading mathematician in Berlin around the turn of the century; the work in which the inequality appears is devoted to a question about minimal surfaces.

EXAMPLE 6 Verify the positivity property D4 of Theorem 1.3.

SOLUTION We let $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, and we find that

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2.$$

A sum of squares is nonnegative and can be zero if and only if each summand is zero. But a summand v_i^2 is itself a square, and will be zero if and only if $v_i = 0$. This completes the demonstration. ■

It is important to observe that the norm of a vector can be expressed in terms of its dot product with itself. Namely, for a vector $\mathbf{v}$ in $\mathbb{R}^n$ we have

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}. \tag{11}$$

Letting $\mathbf{v} = [v_1, v_2, \cdots, v_n]$, we have

$$\mathbf{v} \cdot \mathbf{v} = v_1v_1 + v_2v_2 + \cdots + v_nv_n = \|\mathbf{v}\|^2.$$

Equation 11 enables us to use the algebraic properties of the dot product in Theorem 1.3 to prove things about the norm. This technique is illustrated in the proof of the Schwarz and triangle inequalities at the end of this section. Here is another illustration.

EXAMPLE 7 Show that the sum of the squares of the lengths of the diagonals of a parallelogram in $\mathbb{R}^n$ is equal to the sum of the squares of the lengths of the sides. (This is the *parallelogram relation*).

SOLUTION We take our parallelogram with vertex at the origin and with vectors $\mathbf{v}$ and $\mathbf{w}$ emanating from the origin to form two sides, as shown in Figure 1.25. The lengths of the diagonals are then $\|\mathbf{v} + \mathbf{w}\|$ and $\|\mathbf{v} - \mathbf{w}\|$. Using Eq. (11) and properties of the dot product, we have

$$\begin{aligned}
&\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 \\
&\qquad = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) + (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \\
&\qquad = (\mathbf{v} \cdot \mathbf{v}) + 2(\mathbf{v} \cdot \mathbf{w}) + (\mathbf{w} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{v}) - 2(\mathbf{v} \cdot \mathbf{w}) + (\mathbf{w} \cdot \mathbf{w}) \\
&\qquad = 2(\mathbf{v} \cdot \mathbf{v}) + 2(\mathbf{w} \cdot \mathbf{w}) \\
&\qquad = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2,
\end{aligned}$$

which is what we wished to prove. ■

The definition of the angle θ between two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ leads naturally to this definition of perpendicular vectors, or *orthogonal* vectors as they are usually called in linear algebra.

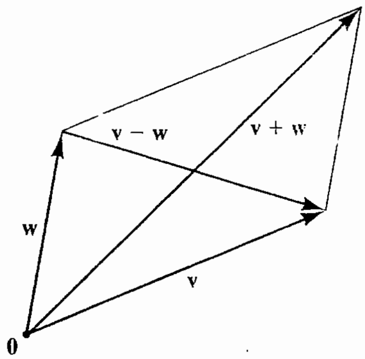

FIGURE 1.25
The parallelogram has v + w and v − w as vector diagonals.

DEFINITION 1.7 Perpendicular or Orthogonal Vectors

> Two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ are **perpendicular** or **orthogonal**, and we write $\mathbf{v} \perp \mathbf{w}$, if $\mathbf{v} \cdot \mathbf{w} = 0$.

EXAMPLE 8 Determine whether the vectors $\mathbf{v} = [4, 1, -2, 1]$ and $\mathbf{w} = [3, -4, 2, -4]$ are perpendicular.

SOLUTION We have

$$\mathbf{v} \cdot \mathbf{w} = (4)(3) + (1)(-4) + (-2)(2) + (1)(-4) = 0.$$

Thus, $\mathbf{v} \perp \mathbf{w}$. ∎

Application to Velocity Vectors and Navigation

The next two examples are concerned with another important physical vector model. A vector is the **velocity vector** of a moving object at an instant if it points in the direction of the motion and if its magnitude is the speed of the object at that instant. Physicists tell us that if a boat cruising with a heading and speed that would give it a still-water velocity vector $\mathbf{s}$ is also subject to a current that has velocity vector $\mathbf{c}$, then the actual velocity vector of the boat is $\mathbf{v} = \mathbf{s} + \mathbf{c}$.

EXAMPLE 9 Suppose that a ketch is sailing at 8 knots, following a course of 010° (that is, 10° east of north), on a bay that has a 2-knot current setting in the direction 070° (that is, 70° east of north). Find the course and speed made good. (The expression *made good* is standard navigation terminology for the actual course and speed of a vessel over the bottom.)

SOLUTION The velocity vectors s for the ketch and c for the current are shown in Figure 1.26, in which the vertical axis points due north. We find s and c by using a calculator and computing

$$\mathbf{s} = [8 \cos 80°, 8 \sin 80°] \approx [1.39, 7.88]$$

and

$$\mathbf{c} = [2 \cos 20°, 2 \sin 20°] \approx [1.88, 0.684].$$

By adding s and c, we find the vector v representing the course and speed of the ketch over the bottom—that is, the course and speed made good. Thus we have $\mathbf{v} = \mathbf{s} + \mathbf{c} \approx [3.27, 8.56]$. Therefore, the speed of the ketch is

$$\|\mathbf{v}\| \approx \sqrt{(3.27)^2 + (8.56)^2} \approx 9.16 \text{ knots},$$

and the course made good is given by

$$90° - \arctan\left(\frac{8.56}{3.27}\right) \approx 90° - 69° = 21°.$$

That is, the course is 021°. ■

EXAMPLE 10 Suppose the captain of our ketch realizes the importance of keeping track of the current. He wishes to sail in 5 hours to a harbor that bears 120° and is 35 nautical miles away. That is, he wishes to make good the course 120° and the speed 7 knots. He knows from a tide and current table that the current is setting due south at 3 knots. What should be his course and speed through the water?

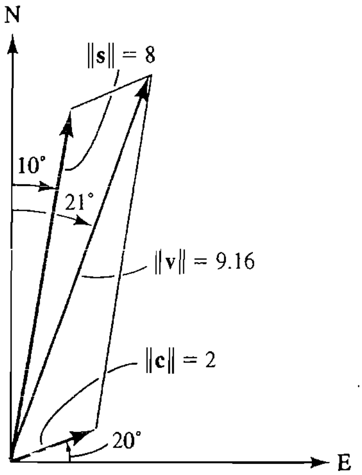

FIGURE 1.26
The vector v = s + c.

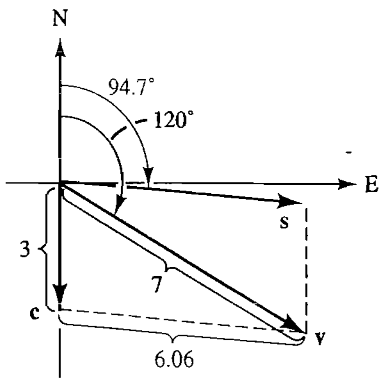

FIGURE 1.27
The vector s = v − c.

SOLUTION In a vector diagram (see Figure 1.27), we again represent the course and speed to be made good by a vector $\mathbf{v}$ and the velocity of the current by $\mathbf{c}$. The correct course and speed to follow are represented by the vector $\mathbf{s}$, which is obtained by computing

$$\begin{aligned}\mathbf{s} &= \mathbf{v} - \mathbf{c} \\ &= [7 \cos 30°, -7 \sin 30°] - [0, -3] \\ &\approx [6.06, -3.5] - [0, -3] = [6.06, -0.5].\end{aligned}$$

Thus the captain should steer course $90° - \arctan(-0.5/6.06) \approx 90° + 4.7° = 94.7°$ and should proceed at

$$\|\mathbf{s}\| \approx \sqrt{(6.06)^2 + (-0.5)^2} \approx 6.08 \text{ knots.}$$ ∎

Proofs of the Schwarz and Triangle Inequalities

The proofs of the Schwarz and triangle inequalities illustrate the use of algebraic properties of the dot product in proving properties of the norm. Recall Eq. (11): for a vector $\mathbf{v}$ in $\mathbb{R}^n$, we have

$$\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

THEOREM 1.4 Schwarz Inequality

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Then $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \, \|\mathbf{w}\|$.

PROOF Because the norm of a vector is a real number and the square of a real number is nonnegative, for any scalars r and s we have

$$\|r\mathbf{v} + s\mathbf{w}\|^2 \geq 0. \tag{12}$$

Using relation (11), we find that

$$\begin{aligned}\|r\mathbf{v} + s\mathbf{w}\|^2 &= (r\mathbf{v} + s\mathbf{w}) \cdot (r\mathbf{v} + s\mathbf{w}) \\ &= r^2(\mathbf{v} \cdot \mathbf{v}) + 2rs(\mathbf{v} \cdot \mathbf{w}) + s^2(\mathbf{w} \cdot \mathbf{w}) \geq 0\end{aligned}$$

for all choices of scalars r and s. Setting $r = \mathbf{w} \cdot \mathbf{w}$ and $s = -(\mathbf{v} \cdot \mathbf{w})$, the preceding inequality becomes

$$\begin{aligned}&(\mathbf{w} \cdot \mathbf{w})^2(\mathbf{v} \cdot \mathbf{v}) - 2(\mathbf{w} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{w})^2 + (\mathbf{v} \cdot \mathbf{w})^2(\mathbf{w} \cdot \mathbf{w}) \\ &\quad = (\mathbf{w} \cdot \mathbf{w})^2(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{w} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{w})^2 \geq 0.\end{aligned}$$

Factoring out $(\mathbf{w} \cdot \mathbf{w})$, we see that

$$(\mathbf{w} \cdot \mathbf{w})[(\mathbf{w} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{w})^2] \geq 0. \tag{13}$$

If $\mathbf{w} \cdot \mathbf{w} = 0$, then $\mathbf{w} = \mathbf{0}$ by the positivity property in Theorem 1.3, and the Schwarz inequality is then true because it reduces to $0 \leq 0$. If $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w} \neq 0$, then the expression in square brackets in relation (13) must also be nonnegative—that is,

$$(\mathbf{w} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{v} \cdot \mathbf{w})^2 \geq 0,$$

and so

$$(\mathbf{v} \cdot \mathbf{w})^2 \leq (\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) = \|\mathbf{v}\|^2\|\mathbf{w}\|^2.$$

Taking square roots, we obtain the Schwarz inequality. ▲

The Schwarz inequality can be used to prove the triangle inequality that was illustrated in Figure 1.22.

THEOREM 1.5 The Triangle Inequality

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbb{R}^n$. Then $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

PROOF Using properties of the dot product, as well as the Schwarz inequality, we have

$$\begin{aligned} \|\mathbf{v} + \mathbf{w}\|^2 &= (\mathbf{v} + \mathbf{w})\cdot(\mathbf{v} + \mathbf{w}) \\ &= (\mathbf{v} \cdot \mathbf{v}) + 2(\mathbf{v} \cdot \mathbf{w}) + (\mathbf{w} \cdot \mathbf{w}) \\ &\leq (\mathbf{v} \cdot \mathbf{v}) + 2\|\mathbf{v}\|\,\|\mathbf{w}\| + (\mathbf{w} \cdot \mathbf{w}) \\ &= \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\,\|\mathbf{w}\| + \|\mathbf{w}\|^2 \\ &= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2. \end{aligned}$$

The desired relation follows at once, by taking square roots. ▲

SUMMARY

Let $\mathbf{v} = [v_1, v_2, \ldots, v_n]$ *and* $\mathbf{w} = [w_1, w_2, \ldots, w_n]$ *be vectors in* $\mathbb{R}^n$.

1. The *norm* or *magnitude* of $\mathbf{v}$ is $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.
2. The norm satisfies the properties given in Theorem 1.2.
3. A *unit vector* is a vector of magnitude 1.
4. The *dot product* of $\mathbf{v}$ and $\mathbf{w}$ is $\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$.
5. The dot product satisfies the properties given in Theorem 1.3.
6. Moreover, we have $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$ and $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\|\,\|\mathbf{w}\|$ (*Schwarz inequality*), and also $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ (*triangle inequality*).
7. The *angle* θ between the vectors $\mathbf{v}$ and $\mathbf{w}$ can be found by using the relation $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\|\,\|\mathbf{w}\|(\cos\theta)$.
8. The vectors $\mathbf{v}$ and $\mathbf{w}$ are *orthogonal* (*perpendicular*) if $\mathbf{v} \cdot \mathbf{w} = 0$.