M6. Enter **x** = **a** and enter **y** = **b**. Then enter

```
anglxy = acos(sum(x .* y)/(norm(x)*norm(y)))
```

to compute the angle (in radians) between **a** and **b**. You should study this formula until you understand why it provides the angle between **a** and **b**.

M7. Compute the angle between **b** and **c** using the technique suggested in Exercise M2. Namely, enter **x** = **b**, enter **y** = **c**, and then use the upward arrow until the cursor is at the formula for **anglxy** and press Enter.

M8. Move the cursor to the formula for **anglxy** and edit the formula so that the angle will be given in degrees rather than in radians. Recall that we multiply by $180/\pi$ to convert radians to degrees. The number π is available as **pi** in MATLAB. Check your editing by computing the angle between the vectors [1, 0] and [0, 1]. Then find the angle between **u** and **w** in degrees.

M9. Find the angle between 3**u** − 2**w** and 4**v** + 2**w** in degrees.

1.3 MATRICES AND THEIR ALGEBRA

The Notation $A\mathbf{x} = \mathbf{b}$

We saw in Section 1.1 that we can write a linear system such as

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ 3x_1 + 5x_2 &= 19 \end{aligned} \tag{1}$$

in the unknowns x_1 and x_2 as a single column vector equation—namely,

$$x_1\begin{bmatrix}1\\3\end{bmatrix} + x_2\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}-1\\19\end{bmatrix}. \tag{2}$$

Another useful way to abbreviate this linear system is

$$\underset{A}{\begin{bmatrix}1 & -2\\3 & 5\end{bmatrix}}\underset{\mathbf{x}}{\begin{bmatrix}x_1\\x_2\end{bmatrix}} = \underset{\mathbf{b}}{\begin{bmatrix}-1\\19\end{bmatrix}}. \tag{3}$$

Let us denote by A the bracketed array on the left containing the coefficients of the linear system. This array A is followed by the column vector **x** of unknowns, and let the column vector of constants after the equal sign be denoted by **b**. We can then symbolize the linear system as

$$A\mathbf{x} = \mathbf{b}. \tag{4}$$

There are several reasons why notation (4) is a convenient way to write a linear system. It is much easier to denote a general linear system by $A\mathbf{x} = \mathbf{b}$ than to write out several linear equations with unknowns $x_1, x_2, \ldots, x_n$, subscripted letters for the coefficients of the unknowns, and constants $b_1, b_2, \ldots, b_m$ to the right of the equal signs. [Just look ahead at Eq. (1) on page 54.] Also, a single linear equation in just one unknown can be written in the form $ax = b$ ($2x = 6$, for example), and the notation $A\mathbf{x} = \mathbf{b}$ is suggestively similar. Furthermore, we will see in Section 2.3 that we can regard such an array A as defining a *function* whose value at $\mathbf{x}$ we will write as $A\mathbf{x}$, much as we write $\sin x$. Solving a linear system $A\mathbf{x} = \mathbf{b}$ can thus be regarded as finding the vector $\mathbf{x}$ such that this function applied to $\mathbf{x}$ yields the vector $\mathbf{b}$. For all of these reasons, the notation $A\mathbf{x} = \mathbf{b}$ for a linear system is one of the most useful notations in mathematics.

It is very important to remember that

> $A\mathbf{x}$ is equal to a linear combination of the *column vectors* of A,

as illustrated by Eqs. (2) and (3)—namely,

$$\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix}. \tag{5}$$

The Notion of a Matrix

We now introduce the usual terminology and notation for an array of numbers such as the coefficient array A in Eq. (3).

A *matrix* is an ordered rectangular array of numbers, usually enclosed in parentheses or square brackets. For example,

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & -6 \\ -3 & -1 & -1 \end{bmatrix}$$

are matrices. We will generally use upper-case letters to denote matrices.

The size of a matrix is specified by the number of (horizontal) rows and the number of (vertical) columns that it contains. The matrix A above contains two rows and two columns and is called a 2×2 (read "2 by 2") matrix. Similarly, B is a 4×3 matrix. In writing the notation $m \times n$ to describe the shape of a matrix, we always write the number of rows first. An $n \times n$ matrix has the same number of rows as columns and is said to be a *square matrix*. We recognize that a $1 \times n$ matrix is a row vector with n components, and an $m \times 1$ matrix is a column vector with m components. The rows of a matrix are its *row vectors* and the columns are its *column vectors*.

Double subscripts are commonly used to indicate the location of an entry in a matrix that is not a row or column vector. The first subscript gives the number of the row in which the entry appears (counting from the top), and the second subscript gives the number of the column (counting from the left). Thus an $m \times n$ matrix A may be written as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & & \vdots & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

If we want to express the matrix B on page 36 as $[b_{ij}]$, we would have $b_{11} = -1$, $b_{21} = 2$, $b_{32} = 5$, and so on.

Matrix Multiplication

We are going to consider the expression $A\mathbf{x}$ shown in Eq. (3) to be the *product* of the matrix A and the column vector $\mathbf{x}$. Looking back at Eq. (5), we see that such a product of a matrix A with a column vector $\mathbf{x}$ should be the linear combination of the column vectors of A having as coefficients the components in the vector $\mathbf{x}$. Here is a *nonsquare* example in which we replace the vector $\mathbf{x}$ of unknowns by a specific vector of numbers.

EXAMPLE 1 Write as a linear combination and then compute the product

$$\begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & -7 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 8 \end{bmatrix}.$$

HISTORICAL NOTE THE TERM *MATRIX* is first mentioned in mathematical literature in an 1850 paper of James Joseph Sylvester (1814–1897). The standard nontechnical meaning of this term is "a place in which something is bred, produced, or developed." For Sylvester, then, a matrix, which was an "oblong arrangement of terms," was an entity out of which one could form various square pieces to produce determinants. These latter quantities, formed from square matrices, were quite well known by this time.

James Sylvester (his original name was James Joseph) was born into a Jewish family in London, and was to become one of the supreme algebraists of the nineteenth century. Despite having studied for several years at Cambridge University, he was not permitted to take his degree there because he "professed the faith in which the founder of Christianity was educated." Therefore, he received his degrees from Trinity College, Dublin. In 1841 he accepted a professorship at the University of Virginia; he remained there only a short time, however, his horror of slavery preventing him from fitting into the academic community. In 1871 he returned to the United States to accept the chair of mathematics at the newly opened Johns Hopkins University. In between these sojourns, he spent about 10 years as an attorney, during which time he met Arthur Cayley (see the note on p. 3), and 15 years as Professor of Mathematics at the Royal Military Academy, Woolwich. Sylvester was an avid poet, prefacing many of his mathematical papers with examples of his work. His most renowned example was the "Rosalind" poem, a 400-line epic, each line of which rhymed with "Rosalind."

SOLUTION Using Eq. (5) as a guide, we find that

$$\begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & -7 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \\ 8 \end{bmatrix} = -2\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 5\begin{bmatrix} -3 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} 21 \\ -34 \end{bmatrix}.$$

■

Note that in Example 1, the first entry 21 of the final column vector is computed as $(-2)(2) + (5)(-3) + (8)(5)$, which is precisely the dot product of the first row vector $[2 \ -3 \ 5]$ of the matrix with the column vector $\begin{bmatrix} -2 \\ 5 \\ 8 \end{bmatrix}$. Similarly, the second component -34 of our answer is the dot product of the second row vector $[-1 \ 4 \ -7]$ with this column vector.

In a similar fashion, we see that the ith component of a column vector $A\mathbf{b}$ will be equal to the dot product of the ith row of A with the column vector $\mathbf{b}$. We should also note from Example 1 that the number of components in a row of A will have to be equal to the number of components in the column vector $\mathbf{b}$ if we are to compute the product $A\mathbf{b}$.

We have illustrated how to compute a product $A\mathbf{b}$ of an $m \times n$ matrix with an $n \times 1$ column vector. We can extend this notion to a product AB of an $m \times n$ matrix A with an $n \times s$ matrix B.

> The product AB is the matrix whose jth column is the product of A with the jth column vector of B.

Letting $\mathbf{b}_j$ be the jth column vector of B, we write $AB = C$ symbolically as

$$A\underbrace{\begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_s \\ | & | & & | \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} | & | & & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_s \\ | & | & & | \end{bmatrix}}_{C}.$$

Because B has s columns, C has s columns. The comments after Example 1 indicate that the ith entry in the jth column of AB is the dot product of the ith row of A with the jth column of B. We give a formal definition.

DEFINITION 1.8 Matrix Multiplication

Let $A = [a_{ik}]$ be an $m \times n$ matrix, and let $B = [b_{kj}]$ be an $n \times s$ matrix. The **matrix product** AB is the $m \times s$ matrix $C = [c_{ij}]$, where c_{ij} is the dot product of the ith row vector of A and the jth column vector of B.

We illustrate the choice of row i from A and column j from B to find the element c_{ij} in AB, according to Definition 1.8, by the equation

$$AB = [c_{ij}] = \begin{bmatrix} a_{11} & — & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & — & a_{in} \\ \vdots & & \vdots \\ a_{m1} & — & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1s} \\ | & & | & & | \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{ns} \end{bmatrix},$$

where

$$c_{ij} = (i\text{th row vector of } A) \cdot (j\text{th column vector of } B).$$

In summation notation, we have

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= \sum_{k=1}^{n} a_{ik}b_{kj}. \end{aligned} \tag{6}$$

Notice again that AB is defined only when the second size-number (the number of columns) of A is the same as the first size-number (the number of rows) of B. The product matrix has the shape

$$(\text{First size-number of } A) \times (\text{Second size-number of } B).$$

EXAMPLE 2 Let A be a 2×3 matrix, and let B be a 3×5 matrix. Find the sizes of AB and BA, if they are defined.

SOLUTION Because the second size-number, 3, of A equals the first size-number, 3, of B, we see that AB is defined; it is a 2×5 matrix. However, BA is not defined, because the second size-number, 5, of B is not the same as the first size-number, 2, of A. ■

EXAMPLE 3 Compute the product

$$\begin{bmatrix} -2 & 3 & 2 \\ 4 & 6 & -2 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 & 5 \\ 3 & 0 & 1 & 1 \\ -2 & 3 & 5 & -3 \end{bmatrix}.$$

SOLUTION The product is defined, because the left-hand matrix is 2×3 and the right-hand matrix is 3×4; the product will have size 2×4. The entry in the first row and first column position of the product is obtained by taking the dot product of the first row vector of the left-hand matrix and the first column vector of the right-hand matrix, as follows:

$$(-2)(4) + (3)(3) + (2)(-2) = -8 + 9 - 4 = -3.$$

The entry in the second row and third column of the product is the dot product of the second row vector of the left-hand matrix and the third column vector of the right-hand one:

$$(4)(2) + (6)(1) + (-2)(5) = 8 + 6 - 10 = 4,$$

and so on, through the remaining row and column positions of the product. Eight such computations show that

$$\begin{bmatrix} -2 & 3 & 2 \\ 4 & 6 & -2 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 & 5 \\ 3 & 0 & 1 & 1 \\ -2 & 3 & 5 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 8 & 9 & -13 \\ 38 & -10 & 4 & 32 \end{bmatrix}.$$

∎

Examples 2 and 3 show that sometimes AB is defined when BA is not. Even if both AB and BA are defined, however, it need not be true that $AB = BA$:

> Matrix multiplication is not commutative.

EXAMPLE 4 Let

$$A = \begin{bmatrix} 0 & 2 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix}.$$

Compute AB and BA.

HISTORICAL NOTE MATRIX MULTIPLICATION originated in the composition of linear substitutions, fully explored by Carl Friedrich Gauss (1777–1855) in his *Disquisitiones Arithmeticae* of 1801 in connection with his study of quadratic forms. Namely, if $F = Ax^2 + 2Bxy + Cy^2$ is such a form, then the linear substitution

$$x = ax' + by' \qquad y = cx' + dy' \tag{i}$$

transforms F into a new form F' in the variables x' and y'. If a second substitution

$$x' = ex'' + fy'' \qquad y' = gx'' + hy'' \tag{ii}$$

transforms F' into a form F'' in x'', y'', then the composition of the substitutions, found by replacing x', y' in (i) by their values in (ii), gives a substitution transforming F into F'':

$$x = (ae + bg)x'' + (af + bh)y'' \qquad y = (ce + dg)x'' + (cf + dh)y''. \tag{iii}$$

The coefficient matrix of substitution (iii) is the product of the coefficient matrices of substitutions (i) and (ii). Gauss performed an analogous computation in his study of substitutions in forms in three variables, which produced the rule for multiplication of 3×3 matrices.

Gauss, however, did not explicitly refer to this idea of composition as a "multiplication." That was done by his student Ferdinand Gotthold Eisenstein (1823–1852), who introduced the notation $S \times T$ to denote the substitution composed of S and T. About this notation Eisenstein wrote, "An algorithm for calculation can be based on this; it consists of applying the usual rules for the operations of multiplication, division, and exponentiation to symbolical equations between linear systems; correct symbolical equations are always obtained, the sole consideration being that the order of the factors may not be altered."

SOLUTION We compute that

$$AB = \begin{bmatrix} 4 & 10 \\ 10 & 28 \end{bmatrix}, \quad \text{and} \quad BA = \begin{bmatrix} 3 & 5 \\ 15 & 29 \end{bmatrix}.$$

■

Of course, for a square matrix A, we denote AA by A^2, AAA by A^3, and so on. It can be shown that matrix multiplication is associative; that is,

$$A(BC) = (AB)C$$

whenever the product is defined. This is not difficult to prove from the definition, although keeping track of subscripts can be a bit challenging. We leave the proof as Exercise 33, whose solution is given in the back of this text.

The $n \times n$ Identity Matrix

Let I be the $n \times n$ matrix $[a_{ij}]$ such that $a_{ii} = 1$ for $i = 1, \ldots, n$ and $a_{ij} = 0$ for $i \neq j$. That is,

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ & 1 & & \text{\huge 0} & \\ & & 1 & & \\ & \text{\huge 0} & & \ddots & \\ & & & & 1 \end{bmatrix},$$

where the large zeros above and below the diagonal in the second matrix indicate that each entry of the matrix in those positions is 0. If A is any $m \times n$ matrix and B is any $n \times s$ matrix, we can show that

$$AI = A \quad \text{and} \quad IB = B.$$

We can understand why this is so if we think about why it is that

$$\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 7 \end{bmatrix}.$$

Because of the relations $AI = A$ and $IB = B$, the matrix I is called the $n \times n$ **identity matrix**. It behaves for multiplication of $n \times n$ matrices exactly as the scalar 1 behaves for multiplication of scalars. We have one such square identity matrix for each integer 1, 2, 3, ... To keep notation simple, we denote them all by I, rather than by $I_1, I_2, I_3, \ldots$. The size of I will be clear from the context.

The identity matrix is an example of a **diagonal matrix**—namely, a square matrix with zero entries except possibly on the **main diagonal**, which extends from the upper left corner to lower right corner.

Other Matrix Operations

Although multiplication is a very important matrix operation for our work, we will have occasion to add and subtract matrices, and to multiply a matrix by a scalar, in later chapters. Matrix addition, subtraction, and scalar multiplication are natural extensions of these same operations for vectors as defined in Section 1.1; they are again performed on entries in corresponding positions.

DEFINITION 1.9 Matrix Addition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same size $m \times n$. The **sum** $A + B$ of these two matrices is the $m \times n$ matrix $C = [c_{ij}]$, where

$$c_{ij} = a_{ij} + b_{ij}.$$

That is, the sum of two matrices of the same size is the matrix of that size obtained by adding corresponding entries.

EXAMPLE 5 Find

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 3 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 2 \\ 1 & -5 & 3 \end{bmatrix}.$$

SOLUTION The sum is the matrix

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & -2 & 2 \end{bmatrix}.$$

■

EXAMPLE 6 Find

$$\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 4 & 6 \\ 3 & 7 & -1 \end{bmatrix}.$$

SOLUTION The sum is undefined, because the matrices are not the same size. ■

Let A be an $m \times n$ matrix, and let O be the $m \times n$ matrix all of whose entries are zero. Then,

$$A + O = O + A = A.$$

The matrix O is called the $m \times n$ **zero matrix**; the size of such a zero matrix is made clear by the context.

DEFINITION 1.10 Scalar Multiplication

Let $A = [a_{ij}]$, and let r be a scalar. The **product** rA of the scalar r and the matrix A is the matrix $B = [b_{ij}]$ having the same size as A, where

$$b_{ij} = ra_{ij}.$$

EXAMPLE 7 Find

$$2\begin{bmatrix} -2 & 1 \\ 3 & -5 \end{bmatrix}.$$

SOLUTION Multiplying each entry of the matrix by 2, we obtain the matrix

$$\begin{bmatrix} -4 & 2 \\ 6 & -10 \end{bmatrix}.$$ ∎

For matrices A and B of the same size, we define the **difference** $A - B$ to be

$$A - B = A + (-1)B.$$

The entries in $A - B$ are obtained by subtracting the entries of B from entries in the corresponding positions in A.

EXAMPLE 8 If

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & -5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 5 \\ 4 & -2 & 1 \end{bmatrix},$$

find $2A - 3B$.

SOLUTION We find that

$$2A - 3B = \begin{bmatrix} 9 & -2 & -7 \\ -12 & 10 & -13 \end{bmatrix}.$$ ∎

We introduced the *transpose operation* to change a row vector to a column vector, or vice versa, in Section 1.1. We generalize this operation for application to matrices, changing all the row vectors to column vectors, which results in all the column vectors becoming row vectors.

DEFINITION 1.11 Transpose of a Matrix; Symmetric Matrix

The matrix B is the **transpose** of the matrix A, written $B = A^T$, if each entry b_{ij} in B is the same as the entry a_{ji} in A, and conversely. If A is a matrix and if $A = A^T$, then the matrix A is **symmetric**.

EXAMPLE 9 Find A^T if

$$A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & 2 & 7 \end{bmatrix}.$$

SOLUTION We have

$$A^T = \begin{bmatrix} 1 & -3 \\ 4 & 2 \\ 5 & 7 \end{bmatrix}.$$

Notice that the rows of A become the columns of A^T. ∎

A symmetric matrix must be square. Symmetric matrices arise in some applications, as we shall see in Chapter 8.

EXAMPLE 10 Fill in the missing entries in the 4×4 matrix

$$\begin{bmatrix} 5 & -6 & & 8 \\ & 3 & & \\ -2 & 1 & 0 & 4 \\ & 11 & & -1 \end{bmatrix}$$

to make it symmetric.

SOLUTION Because rows must match corresponding columns, we obtain

$$\begin{bmatrix} 5 & -6 & -2 & 8 \\ -6 & 3 & 1 & 11 \\ -2 & 1 & 0 & 4 \\ 8 & 11 & 4 & -1 \end{bmatrix}.$$

■

In Example 10, note the symmetry in the main diagonal.

We have explained that we will often regard vectors in $\mathbb{R}^n$ as column vectors. If **a** and **b** are two column vectors in $\mathbb{R}^n$, the dot product $\mathbf{a} \cdot \mathbf{b}$ can be written in terms of the transpose operation and matrix multiplication—namely,

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T\mathbf{b} = [a_1 \; a_2 \; \cdots \; a_n] \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}. \tag{7}$$

Strictly speaking, $\mathbf{a}^T\mathbf{b}$ is a 1×1 matrix, and its sole entry is $\mathbf{a} \cdot \mathbf{b}$. Identifying a 1×1 matrix with its sole entry should cause no difficulty. The use of Eq. (7) makes some formulas given later in the text much easier to handle.

Properties of Matrix Operations

For handy reference, we box the properties of matrix algebra and of the transpose operation. These properties are valid for all vectors, scalars, and matrices for which the indicated quantities are defined. The exercises ask for proofs of most of them. The proofs of the properties of matrix algebra not involving matrix multiplication are essentially the same as the proofs of the same properties presented for vector algebra in Section 1.1. We would expect this because those operations are performed just on corresponding entries, and every vector can be regarded as either a $1 \times n$ or an $n \times 1$ matrix.

Properties of Matrix Algebra

$A + B = B + A$	**Commutative law of addition**
$(A + B) + C = A + (B + C)$	**Associative law of addition**
$A + O = O + A = A$	**Identity for addition**
$r(A + B) = rA + rB$	**A left distributive law**
$(r + s)A = rA + sA$	**A right distributive law**
$(rs)A = r(sA)$	**Associative law of scalar multiplication**
$(rA)B = A(rB) = r(AB)$	**Scalars pull through**
$A(BC) = (AB)C$	**Associative law of matrix multiplication**
$IA = A$ and $BI = B$	**Identity for matrix multiplication**
$A(B + C) = AB + AC$	**A left distributive law**
$(A + B)C = AC + BC$	**A right distributive law**

Properties of the Transpose Operation

$(A^T)^T = A$	**Transpose of the transpose**
$(A + B)^T = A^T + B^T$	**Transpose of a sum**
$(AB)^T = B^TA^T$	**Transpose of a product**

EXAMPLE 11 Prove that $A(B + C) = AB + AC$ for any $m \times n$ matrix A and any $n \times s$ matrices B and C.

SOLUTION Let $A = [a_{ij}]$, $B = [b_{jk}]$ and $C = [c_{jk}]$. Note the use of j, which runs from 1 to n, as both the second index for entries in A and the first index for the entries in B and C. The entry in the ith row and kth column of $A(B + C)$ is

$$\sum_{j=1}^{n} a_{ij}(b_{jk} + c_{jk}).$$

By familiar properties of real numbers, this sum is also equal to

$$\sum_{j=1}^{n} (a_{ij}b_{jk} + a_{ij}c_{jk}) = \sum_{j=1}^{n} a_{ij}b_{jk} + \sum_{j=1}^{n} a_{ij}c_{jk}$$

which we recognize as the sum of the entries in the ith row and kth columns of the matrices AB and AC. This completes the proof. ∎