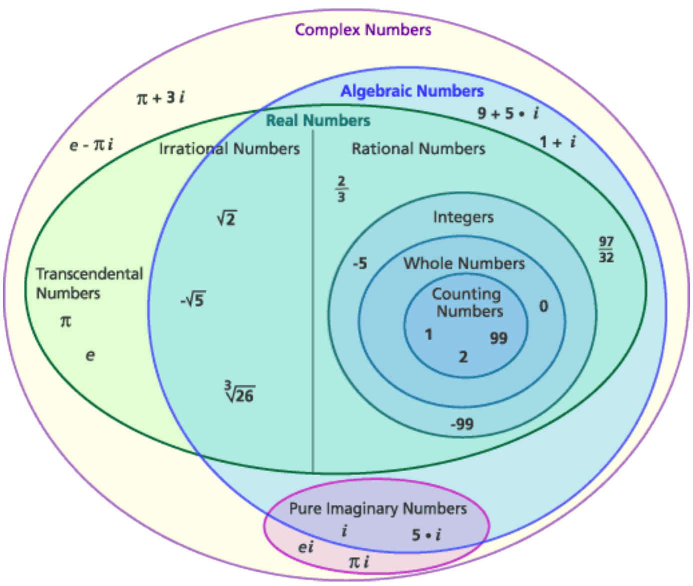
第四組報告 Transcendental Number 超越數

組員:410931130 李簡奕辰
410731151 謝少然
410831227 張湞昇
410831149 楊弘暐
410631244 沈致均

前言

某次意外參加某校單車節，並參加該校數學系介紹，當初學習內容是數學歸納法與超越數，也介紹很表面的內容，讓高三生要填寫志願時，對數學系有進一步認識，知道這科系都在做些什麼事情。也因為介紹了很表面，令我對它 產生了好奇，想一探究竟，剛好藉這次機會多了解超越數，於是將這次小組小論文主題定為超越數。

從古希臘幾何三大問題，方圓問題、倍立方問題、三等分角問題經過時間推進，數學家如何將幾何問題轉化為代數問題並引入超越數，這想法去證明。
為什麼 n 次方程有 n 個根，什麼是代數數,什麼是超越數?並講述為何 π 和 e 都是超越數，以及代數基本定理證明等去做說明
數學是科學的皇后，數論是數學的皇后



無理數 Irrational Numbers

假設$\sqrt{2}$是有理數並令$\sqrt{2}=\frac{p}{q}$ *且*(p,q)=1

兩邊平方，得到 $2=\frac{p^2}{q^2}$

將此式改寫成 $2q^2 = p^2$ ，意味p^2為偶數

∵ 平方能保持奇偶性

∴ p 只能為偶數

∴ p^2為偶數

設 p=$2p_1$其中p_1為整數

代入$q^2 = 2{p_1}^2$

同理得知 q 也是偶數

這與(p,q)=1(∋∈)

∴ $\sqrt{2}$是有理數的假設不成立，即無理數

規矩數

定義$q_n z^n + \cdots + q_1 z + q_0=0$，其中$q_i$*為整數且*$q_n \neq 0$

$$-4x^4 + 6x^2 - 1 = 0$$

$$X=\frac{1}{2}\sqrt{3+\sqrt{5}}$$

$$x^3 - 2 = 0$$

$$x = \sqrt[3]{2}$$

但尺規作圖無法開三次方，所以$\sqrt[3]{2}$是代數數不是規矩數

代數數 Algebraic Numbers

代數數是代數與數論中的重要概念，指任何整係數多項式的複根。

代數數可以定義為「有理係數多項式的複根」或「整係數多項式的複根」

設 z 為複數。

如果存在正整數 n，以及 n+1 個有理數$q_0, q_1 \ldots q_n$，並且$q_n \neq 0$，

使得：

$$q_n z^n + \cdots + q_1 z + q_0=0$$

則稱 z 是一個代數數。

代數數不一定是實數，實數也不一定是代數數。
代數數的集合是可數的。

$$\text{實數} = \text{有理數} \bigcup \text{無理數}$$

$$\text{複數} = \text{代數數} \bigcup \text{超越數}$$

$$\text{無理數} = \text{無理數中的代數數} \bigcup \text{實數中的超越數}$$

$$\text{實數的代數數} = \text{有理數} \bigcup \text{無理數中的代數數}$$

代數數可數 Algebraic Numbers are Countable

思路:要證明代數數是可數的，就是要證明整係數多項式是可數的
1.證明整係數多項式可數
2.證明代數數可數
因為是集合對應集合 所以是映射(mapping)

假設P_n為 n 次多項式($\deg(p) = n$) 集合，從P_n到正整數 N 的映射($f: P_n \to N$)

$$f(\, a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) = 2^{f(a_0)} 3^{f(a_1)} 5^{f(a_2)} \cdots p(n-1)^{f(a_n)}$$

其中p_n為正整數到質數的任一 bijection(e.g. $p(n)$*為第*n*個質數*)
f_n是整數到非負整數的任一 bijection
(e.g.當$n \geq 0$，$f(n) = 2n$，*當*$n < 0$，$f(n) = -2n - 1$
∵質因數分解有唯一性，這個映射是 bijection
∴ P_n可數
而所有整係數多項式集合
$\because P = \bigcup_{n \in N} P_n$ 是可數個可數集的聯集
∴ 依然可數

證明代數數可數

∵n 次多項式最多有 n 個根，假設R_p*為多項式*p*的根*(代數基本定理，後面會補充)
∴ R_p有限
代數數 $A = \bigcup_{p \in P} R_p$*為可數個有限集的聯集*
因此，依然可數
利用若 p 則 q，非 q 則非 p。我們知道非代數數(超越數)為不可數

(*)圖片說明該式子

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

⋮	⋮	⋮	⋮
P_n	⋯	3	2

(p 為 2 之後第 n 個質數)

⋮	⋮	⋮
${P_n}^{a_n}$	3^{a_1}	2^{a_0}

超越數 Transcendental Number

超越數（transcendental number）是指任何一個不是代數數的無理數。只要它不是任何一個有理係數代數方程的根，它即是超越數。最著名的超越數是 e 以及 π

幾乎所有的實數和複數都是超越數，這是因為代數數的集合是可數集，而實數和複數的集合是不可數集。

超越數是代數數的相反，即說若x*是一個超越數*，那對任何整數 $a_n, a_{n-1}, \cdots, a_0$ *都滿足* $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \neq 0$ ，$where\ a_n \neq 0$

第一個確認為超越數的數，是於 1844 年劉維爾發現

劉維爾數:$\sum_{k=1}^{\infty} 10^{-k!} = 0.110001000000000000000001000 \cdots$

Liouville's theorem complex analysis

Every bounded , entire function $f(z)$ is constant

Suppose a and b are two points on the complex plane.

Take a as the center of the circle, r is the radius

Pack b inside the circle

$$f(b) - f(a) = \frac{1}{2\pi i} \oint \frac{f(z)}{z-b} dz - \frac{1}{2\pi i} \oint \frac{f(z)}{z-a} dz$$

$$= \frac{1}{2\pi i} \oint \left(\frac{1}{z-b} - \frac{1}{z-a} \right) f(z)\, dz$$

$$= \frac{1}{2\pi i} \oint \frac{b-a}{(z-b)(z-a)} f(z)\, dz$$

$$f(b) - f(a) = \frac{b-a}{2\pi i} \oint \frac{f(z)}{(z-b)(z-a)} dz$$

$$|z-a|=r$$

$$|z-b|=|z-a+a-b|\geq|z-a|-|a-b|=r-|a-b|\geq\frac{r}{2}$$

$$|f(b)-f(a)|=\frac{|b-a|}{|2\pi i|}|\oint\frac{f(z)}{(z-b)(z-a)}dz$$

$$\leq\frac{|b-a|}{2\pi}|\oint\frac{M}{(z-b)(z-a)}dz|, f\ is\ bounded$$

$$=\frac{|b-a|}{2\pi}\frac{M}{(\frac{r}{2})r}2\pi r\cdots(*)$$

代數基本定理(Fundamental Theorem of Algebra)

$$A\ poly.equ'n\ \mathbb{P}(z)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0\ where\ a_k \in\mathbb{C}\ and\ k=0,1,\ldots,n,\ \ a_0\neq0,n\geq1\ has\ a\ sol'n\ in\ \mathbb{C}$$

In other words , $\mathbb{C}$ is algebraically closed.

<Proof by contradiction>

Suppose that $\mathbb{P}$(z) has not sol'n.

i.e. $f(z)=\frac{1}{\mathbb{P}(z)}$is entire function and bounded

According to Liouville's theorem ,$f(z)$is constant

So $\mathbb{P}$(z) is also constant. (∋∈)

p is poly.isn't constant

∴$\mathbb{P}$(z) has sol'n

課程反思

對 entire function 的描述

可導出代數基本定理

代數基本定理也間接說明一個 n 次多項式在複數系會有 n 個根

proof of Lindemann-Weierstrass theorem

For any non-zero natural number n and any algebraic numbers $a_1,\ldots,a_n$, if the set $\{a_1,\ldots,a_n\}$ is linearly independent over Q, then $\{e^{a_1},\ldots,e^{a_n}\}$ is algebraically independent over Q

The Lindemann-Weierstrass theorem generalizes both these two statements and their proofs

$If\ \alpha_{1,\dots,}\ \alpha_n$ are algebraic and distinct, and if $\beta_{1,\dots,}\ \beta_n$ are algebraic and non-zero, then $\beta_1\, e^{\alpha_1} + \cdots \beta_n\, e^{\alpha_n} \neq 0$

Note that the facts that e and π are transcendental follow trivially from this theorem.
For example:If e were algebraic , then e is the root of a poly. $\sum \beta_i x^i\ where\ \beta_i \in \mathbb{Q}$ in contradiction to the theorem.

$$The\ following\ construct\ is\ used\ in\ all\ three\ proofs.$$

$$Suppose\ f(x) is\ a\ real\ polynomial,\ and\ let$$

$$\mathrm{I}(t) = \int_0^t e^{t-x} f(x) dx$$

Using I.B.P we get

$$I(t) = \left(-e^{t-x} f(x)\right) \mid {}^{t}_{0} + \int_0^t e^{t-x} f'(x)\ dx = e^t f(0) - f(t) + + \int_0^t e^{t-x} f'(x)\ dx$$

Continuing, and integrating by parts a total of m=deg f times ,we get

$$I(t) = e^t \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(t) \cdots (1)$$

Where $f^{(j)}(x)\ is\ the\ j^{th}\ derivative\ of\ f$

The proof follows the same general lines as above, but there are additional complexities introduced by the arbitrary α_i. In the proof of the transcendality of π we were able to use facts about the relationship of the exponents in the proof; no such relationship is available to us in this more general setting

Again start by supposing $\beta_1\, e^{\alpha_1} + \cdots \beta_n\, e^{\alpha_n} = 0 \cdots (4)$

$where\ the\ \alpha_i,\ \beta_i$ are as given

Claim we can assume, without loss of generality, that $\beta_i \in \mathbb{Z}$.

For if not, take all the expressions formed by substituting for one or more of the β_i one of its conjugates, and multiply those by the equation above. The result is a new expression of the α_i), but where the coefficients are rational numbers. Clear denominators, proving the claim.

Next, claim we can assume that the α_i are a complete set of conjugates, and that if α_i ,$\alpha_j$are conjugates, then $\beta_i = \beta_j$. To see this, choose an irreducible integral polynomial having $\alpha_{1,\dots,}\ \alpha_n$ as roots; let $\alpha_{n+1,\cdots,}\ \alpha_N$ be the remaining roots, and define $\beta_{n+1=\cdots}\ \beta_n = 0$.Then clearly we have

$$\prod_{\sigma \epsilon S_N} (\ \beta_1\, e^{\alpha_{\sigma(1)}} + \cdots \beta_N\, e^{\alpha_{\sigma(N)}}) = 0$$

(Note the similarity with the proof for π). There are N! factors in this product, so expanding the product, it is a sum of terms of the form $e^{h_1\alpha_1+\cdots+h_N\alpha_N}$
with integral coefficients, and $h_1+\cdots+h_N=N!$.
Clearly the set of all such exponents forms a complete set of conjugates.
By symmetry considerations, we see that the coefficients of two conjugate terms are equal.
Also, the product is not identically zero. To see this, consider the term in the product formed by multiplying together, from each factor, the nonzero terms with the largest exponents in the lexicographic order on $\mathbb{C}$.
Since the α_i are unique (because the polynomial is irreducible), there is only one term with this largest exponent, and it has a nonzero coefficient by construction.
Finally, order the terms so that the conjugates of a particular α_i appear together.
That is, for the remainder of the proof we may assume that
$\beta_1 e^{\alpha_1}+\cdots\beta_n e^{\alpha_n}=0$ with the $\beta_i \epsilon \mathbb{Z}$, and that there are integers
0=$n_0<n_1<\cdots<n_r=n$ chosen so that, for each $0\le t<n$
we have $\alpha_{n_t+1},\ldots\ \alpha_{n_{t+1}}$ form a complete set of conjugates$\beta_{n_t+1}=\beta_{n_t+2}=\cdots=\beta_{n_{t+1}}$
Now since $\alpha_i,\ \beta_i$ are algebraic, we can

choose ι $such\ that$ $\iota_{\alpha_i},\iota_{\beta_i}$are algebraic integers. Let $f_i=\iota^{np}\frac{((x-\alpha_1)\ldots(x-\alpha_n))^p}{(x-\alpha_i)}, 1\le i\le n$ where again p is a (large) prime.
We will develop contradictory estimates for $|J_1\ldots J_n|$,where $J_i=\beta_1 I_i(\alpha_i)+\cdots\beta_n I_i(\alpha_n), 1\le i\le n\ and\ I_i$is the integral associated with $f_i,\ as\ above\ (see\ (1))$
Using equations (1) and (4)
we see that

$$\sum_{k=1}^{n}\beta_k I_i(\alpha_k)$$

$$=\sum_{k=1}^{n}\left(\beta_k e^{\alpha_k}\sum_{j=0}^{np-1}f_i^{(j)}(0)\right)-\sum_{k=1}^{n}\left(\beta_k\sum_{j=0}^{np-1}f_i^{(j)}(\alpha_k)\right)$$

$$=(\sum_{j=0}^{np-1}f_i^{(j)}(0))\left(\sum_{k=1}^{n}\beta_k e^{\alpha_k}\right)-\sum_{k=1}^{n}(\sum_{j=0}^{np-1}f_i^{(j)}(\alpha_k))$$

$$=-\sum_{j=0}^{np-1}\sum_{j=0}^{np-1}(\beta_k f_i^{(j)}(\alpha_k))$$

Arguing similarly to the foregoing proofs, we see that $f_i^{(j)}(\alpha_k)$ is an algebraic integer divisible by p! unless j=p−1 and k=i. In this particular case, we have that

$$f_i^{p-1}(\alpha_i) = \iota^{mp}(p-1)!\prod_{k=1}^{n}(\alpha_i - \alpha_k)^p$$

and so again, if p is large enough, this is divisible by (p−1)! but not by p!. Thus
J_i is a nonzero algebraic integer divisible by (p-1)!but not by p!.
As before, we can prove that $J_i \neq 0$.
J_i can be written as follows:

$$J_i = -\sum_{j=0}^{np-1}\sum_{t=0}^{r-1}\beta_{n_t+1}(f_i\,(j)(\alpha_{n_t+1} + \cdots + \alpha_{n_{t+1}})$$

Note that by construction $f_i(x)$ can be written as a polynomial whose coefficients are polynomials in α_i, and the integral coefficients of those polynomials are integers independent of i

Thus, noting that the αiαi form a complete set of conjugates and using the fundamental theorem on sym
$J_i$is in fact a rational number. But it is an algebraic integer, hence an integer. Thus
$J_{i,} \dots J_n \in \mathbb{Z}$. and it is divisible by $((p-1)!)^n$
Thus $|J_1 \dots J_n| \geq (p-1)!$. But the same estimate as in the previous proofs shows that for each I

$$|J_1| \leq \sum_{k=1}^{n}|\beta_k||I_i(\alpha_k)|\,|I_i(\alpha_k)| \leq \sum_{k=1}^{n}|\beta_k\alpha_k|\,e^{|\alpha k|}F_i(|\alpha_k|)$$

which as before is
$\leq c^p$ for some sufficiently large c. These estimates are again in contradiction, proving the theorem.

lexicographic order

Let A be a set equipped with total order <, and let $A^n = A \cdots A$ be the n-fold Cartesian product of A.
Then the lexicographic order < on $A^n$is defined as follows:
$If\ a = (a_1, \dots, a_n) \in A^n\ , then\ a < b$ if $a_1 < b_1\ or$

$$a_1 = b_1$$
$$\vdots$$
$$a_k = b_k$$
$$a_{k+1} < b_{k+1}\ for\ some\ k = 1, \dots, n-1$$

Corollary(*) $\boldsymbol{If\ \alpha \neq 0}$ *is algebraic, then* $e^{i\alpha}$ *is transcendental.*
<pf>

If it were algebraic, say

$$e^{i\alpha} = \beta$$

Then we have

$$e^{i\alpha} - \beta e^{0} = 0$$

in contradiction to the above theorem since $\alpha \neq 0$

Corollary $\boldsymbol{If\ \alpha \neq 0}$***is*** *algebraic, then* $\cos\alpha$ *and* $\sin\alpha$ *is transcendental.*

<pf>

Recall that $\cos\alpha + i\sin\alpha = e^{i\alpha}$, $which\ is\ transecendental.$

If either $\cos\alpha\ or\ \sin\alpha\ were\ algebraic$, $then\ the\ other\ would\ be\ as\ well$

(and thus their sum would be)

Since $sin^2(\alpha) + cos^2(\alpha) = 1$

Hence both $\cos\alpha\ and\ \sin\alpha\ are\ transcendental.$

Corollary $\boldsymbol{If\ \alpha > 0}$ *is algebraic with* $\alpha \neq 1$ *, then* $\ln\alpha$ *is transcendental.*

<pf>

$$If\ \beta = \ln\alpha\ ,\ then\ e^{\beta} = \alpha$$

By Corollary(*) . Since $\alpha\ is\ algebraic$, $\beta\ can't\ be$.

Statement of the Problem

Let α be a real algebraic number . In the simplest case , the central problem of this chapter can be stated as follows:

Determine how small

$$\delta = \delta\left(\alpha; \frac{p}{q}\right) = |\alpha - \frac{p}{q}|$$

Can be for $p \in \mathbb{Z}, q \epsilon \mathbb{N}$. In particular , one might want to

a)find out how much is possible ,i.e., how close rational numbers can get to α ;

or

b)Find out how much is impossible ,i.e., find a lower bound for δ

Since $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$, it follows that for any $\theta \epsilon \mathbb{R}$

(in particular, for any real $\theta \epsilon A$)

And for any $\varepsilon > 0\ there\ are\ infinitely\ many\ rational\ numbers\ \frac{p}{q}$

Such that

$$\left|\theta - \frac{p}{q}\right| < \varepsilon$$

Thus questions(a) and (b) are trivial unless we impose some additional conditions . But these questions become nontrivial for irrational α if we bound q from above and refine (a) and (b) as follows:

A)

Find a positive non-increasing function $\varphi(x) = \varphi(x, \alpha), x\epsilon\mathbb{N}$,such that the inequality

$$|\alpha - \frac{p}{q}| \le \varphi(q)$$

Has infinitely many sol'n (p,q) with $\mathrm{p}\epsilon\mathbb{Z}, q\epsilon\mathbb{N}$

B)

Find a positive non-increasing function$\psi(x) = \psi(x, \alpha), x\epsilon\mathbb{N}$,such that the inequality

$$|\alpha - \frac{p}{q}| \ge \psi(q)$$

Holds for all $\mathrm{p}\epsilon\mathbb{Z}, q\epsilon\mathbb{N}$ with $\frac{p}{q} \ne \alpha\ (or\ at\ least\ for\ all\ such\ that\ with\ q \ge q_0)$.

Approximation of Algebraic Numbers

<Lemma>

It is easy to get complete answers to questions(A) and (B) for rational α .

Let $\alpha = \frac{a}{b}$, $a\epsilon\mathbb{Z}\ b\epsilon\mathbb{N}, \frac{a}{b} \ne \frac{p}{q}; then$

$$\frac{1}{bq} \le \frac{|aq - bp|}{bq} = \left|\alpha - \frac{p}{q}\right|.$$

Since (a,b)=1,by assumption , it follows that the equation ax-by=1 has infinitely many sol'ns $x, y\epsilon\mathbb{Z}$, and so we can take $\varphi(x) = \psi(x) = \frac{1}{bx}$.

These choices are best possible.

<Def>

Let $\theta\epsilon\mathbb{R}$, $and\ let\ \omega(x) > 0\ be\ a\ function\ on\ \mathbb{N}\ that\ approaches\ zero\ x \to \infty$.

We say that θ has a rational approximation of order$\omega(q)$ if for some

$$c = c\big(\theta, \omega(x)\big)\ the\ inequality$$

$$0 < \left|\theta - \frac{p}{q}\right| < c\omega(q)$$

Holds for infinitely many pair s (p,q) with $\mathrm{p}\epsilon\mathbb{Z}, q\epsilon\mathbb{N}$

Thm(*)

The inequality

$$0 < \left|\theta - \frac{p}{q}\right| < \frac{1}{q^2}$$

Has infinitely many sol'ns for any irrational real number θ .

That is, any irrational real number has a rational approximation of order q^{-2}

Quadratic Irrationalities

$$Let\ \alpha\epsilon A, \deg\alpha = 2. There\ exists\ a\ constant\ c = c(\alpha)\ such\ that$$

$$\left|\alpha - \frac{p}{q}\right| > cq^{-2} = \frac{c}{q^2}.$$

Thus by thm(*) , a real quadratic irrationality has a rational approximation of order q^{-2} ;but, by Quadratic Irrationalities , it has no higher order rational approximation.

隨堂練習

Prove

$$\left|\sqrt[3]{2} - \frac{q}{p}\right| > \frac{1}{10p^3}, \forall p \in N\ and\ \exists q \in Z$$

Answer

Obiviously when $\left|\sqrt[3]{2} - \frac{p}{q}\right| \geq 1$

Therefore , we assume that $\left|\sqrt[3]{2} - \frac{p}{q}\right| < 1$

$$|\left(\sqrt[3]{2}\right)^3 - (\frac{p}{q})^3| = |\left(\sqrt[3]{2} - \frac{p}{q}\right)\left(\sqrt[3]{4} + \sqrt[3]{2}\left(\frac{p}{q}\right) + \left(\frac{p}{q}\right)^2\right)|$$

$$= |\left(\sqrt[3]{2} - \frac{p}{q}\right)\left(\sqrt[3]{2} - \frac{p}{q}\right)^2 - 3\sqrt[3]{2}\left(\sqrt[3]{2} - \frac{p}{q}\right) + 3\sqrt[3]{4})|$$

$$< \left|\left(\sqrt[3]{2} - \frac{p}{q}\right)\right|(1 + 4 + 56)\ \cdots\cdots\left(use\ \sqrt[3]{2} < 1.26\right)$$

$$\leq 10\left|\sqrt[3]{2} - \frac{p}{q}\right|$$

$$\frac{1}{p^3} \leq \left|\frac{2p^3 - q^3}{p^3}\right| = |\left(\sqrt[3]{2}\right)^3 - \left(\frac{p}{q}\right)^3|$$

Liouville‘s theorem 就是在考慮，類似這種無理數與有理數$\frac{p}{q}$差的範圍。

Liouville’s thm.

$$Let\ x\ be\ an\ irrational\ number\ that\ is\ algebraic\ of\ degree\ n.$$

$then\ there\ exists\ a\ constant\ c > 0\ (which\ can\ depend\ on\ x)\ such\ that$

$\left|x - \frac{p}{q}\right| \geq \frac{c}{q^n}$ For every pair $p, q \in \mathbb{Z}\ with\ q \neq 0$

Proof

$Let\ r_1, r_2, \cdots r_k be\ the\ rational\ roots\ of\ a\ polynomial\ P\ of\ degree\ n\ that\ has\ x\ as\ a\ roo$Since x is irr

it does not equal any r_i

Let c_1>0 be the minimum of |x-r_i|

If there are r_i ，$let\ c_1 = 1.$

Now let $\alpha = \frac{p}{q}$ where $\alpha \notin \{r_1, r_2, \cdots r_k\}$.

Then:

$$P(\alpha) \neq 0$$

$$|P(\alpha)| \geq \frac{1}{q^n} as\ P(\alpha)\ is\ a \text{ multiple of } \frac{1}{q^n}$$

$$|P(x) - P(\alpha)| \geq \frac{1}{q^n} because\ P(x) = 0$$

Suppose

$$P(x) = \sum_{k=0}^{n} a_k x^k$$

Then

$$P(x) - P(\alpha) = \sum_{k=0}^{n} a_k x^k - \sum_{k=0}^{n} a_k \alpha^k$$

$$= \sum_{k=0}^{n} a_k (x^k - \alpha^k)$$

$$= \sum_{k=1}^{n} a_k (x^k - \alpha^k) \qquad x^0 - \alpha^0 = 0$$

$$= \sum_{k=1}^{n} a_k (x - \alpha) \sum_{i=0}^{k-1} x^{k-1-i} \alpha^i$$

$$= (x - \alpha) \sum_{k=1}^{n} a_k \sum_{i=0}^{k-1} x^{k-1-i} \alpha^i$$

$$\text{Case 1: If } |x-\alpha| \le 1, \alpha \notin \{r_1, r_2, \cdots r_k\}, then$$
$$|\alpha| - |x| \le |x-\alpha|$$
$$|\alpha| - |x| \le 1$$
$$|\alpha| \le |x| + 1$$

Therefore…

$$|P(x) - P(\alpha)| \le |x-\alpha| \sum_{k=1}^{n} |a_k| \sum_{i=0}^{k-1} |x^{k-1-i}\alpha^i|$$

$$\le |x-\alpha| \sum_{k=1}^{n} |a_k| \sum_{i=0}^{k-1} |x^{k-1-i}(|x|+1)^i|$$

$$\le |x-\alpha| \sum_{k=1}^{n} |a_k x^{k-1}| \sum_{i=0}^{k-1} |x^{k-1}\left(\frac{|x|+1}{x}\right)^i|$$

$$\le |x-\alpha| \sum_{k=1}^{n} |a_k x^{k-1}| \sum_{i=0}^{k-1} |\, x^{k-1}(1+\frac{1}{x})^i|$$

$$\le |x-\alpha| \sum_{k=1}^{n} |a_k x^{k-1}| \frac{\left(1+\frac{1}{x}\right)^k - 1}{\left(1+\frac{1}{x}\right) - 1}$$

$$\le |x-\alpha| \sum_{k=1}^{n} |a_k x^k| (\left(1+\frac{1}{x}\right)^k - 1)$$

$$\le |x-\alpha| \sum_{k=1}^{n} |a_k| ((|x|+1)^k - |x|^k)$$

To summarize:

$$|P(x) - P(\alpha)| \le |x-\alpha| c_x$$

Where:

$$c_x = \sum_{k=1}^{n} |a_k| ((|x|+1)^k - |x|^k)$$

So for such α:

$$|x-\alpha| \ge \frac{|P(x) - P(\alpha)|}{c_x} \ge \frac{1}{c_x q^n}$$
$$Case\ 2: If\ |x-\alpha| > 1, \alpha \notin \{r_1, r_2, \cdots r_k\}\ then:$$
$$|x-\alpha| > 1 \ge \frac{1}{q^n}$$

$$Case\ 3: If\ \alpha \in \{r_1, r_2, \cdots r_k\},\ then:$$

$$|x - \alpha| \geq c_1 \geq \frac{c_1}{q^n}$$

$$c = \begin{cases} \frac{1}{c_x} : |x - \alpha| \leq 1, \alpha \notin \{r_1, r_2, \cdots r_k\} \\ 1 : |x - \alpha| > 1, \alpha \notin \{r_1, r_2, \cdots r_k\} \\ c_1 : \alpha \in \{r_1, r_2, \cdots r_k\} \end{cases}$$

Then:

$$\left|x - \frac{p}{q}\right| \geq \frac{c}{q^n}\ \ for\ \ all\ \frac{p}{q}.$$

Liouville's Number

$$x = \sum_{k=1}^{\infty} \frac{1}{10^{n!}}$$

<pf>

By Comparing test

$$\because \frac{1}{10^{k!}} \leq \frac{1}{10^k}\ \ k = 1,2, \ldots$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{10^{k!}} < \sum_{k=1}^{\infty} \frac{1}{10^k} = \frac{1}{9}$$

$Hence\ \sum_{n=1}^{\infty} \frac{1}{10^{n!}}\ is\ convergent\ series$

$$s_n = \frac{p_n}{q_n} = \sum_{k=1}^{\infty} \frac{1}{10^k}$$

$$q_n = 10^{n!}$$

on the other hand

$$\left|x - \frac{p_n}{q_n}\right| = \sum_{k=n+1}^{\infty} \frac{1}{10^{k!}} < \frac{1}{10^{(n+1)!}}\left(1 + \frac{1}{10} + \frac{1}{100} + \cdots\right)$$

$$= \frac{10}{9} \times \frac{1}{10^{(n+1)!}} = \frac{10}{9 \times 10^{n!}} \times \frac{1}{(10^{n!})^n} < \frac{1}{(10^{n!})^n} < \frac{1}{(q_n)^n}$$

$choose\ s_n = n$

$$x\ is\ transcendental\ number$$

總結

今天人們對於超越數的認識

1934 年，蘇聯數學家蓋爾方德和德國數學家施耐德分別獨立的得出了一個關於超越數的定理，我們今天稱之為“Gelfond–Schneider theorem”。

這個定理徹底的解決了 1900 年希爾伯特 23 個數學問題中的第七個的後半部分。

<命題>

如果a是一個不等於 0 和 1 的代數數，b是無理數

那麼a^b是超越數。

根據這個定理，我們可以輕鬆論證得到例如$2^{\sqrt{2}}$、e^{π} *等都是超越數*

這就是我們目前得到的關於代數數、超越數的最新的認知，到今天為止，仍然不知道$e+\pi$、π^e、$e\pi$ 是不是超越數

也許直到某一天，又有哪位數學家能夠發現或發明一種全新的數學結構，從一個全新的角度再一次認識數，才會給出對於無理數、超越數的更深刻的認識，才能解決上述問題。

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