第六組

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代數主題

- 1. Find all real solutions to the equation 4x^2 – 40[x] + 51 = 0. Here, if x is a real number, then [x] denotes the greatest integer that is less than or equal to x.
- 1.找出所有 4x^2 – 40[x] + 51 = 0 的實數解，如果x是實數，則[x]表示為小於或等於x的最大整數

幾何主題

- 2. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and center on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.
- 2.讓ABC為一個高度為1的正三角形，一個半徑為1的圓且圓心與C皆在AB線段的同側，並沿著線段AB滾動。證明在正三角形內弧長永遠一樣長

數論主題

- 3. Determine all positive integers n with the property that n = (d(n))^2 . Here d(n) denotes the number of positive divisors of n.
- 3.決定所有正整數n有性質n=(d(n))^2。d(n)表示為n的公約數

組合主題

- 4. Suppose a1, a2, . . . , a8 are eight distinct integers from {1, 2, . . . , 16, 17}. Show that there is an integer k > 0 such that the equation ai − aj = k has at least three different solutions. Also, find a specific set of 7 distinct integers from {1, 2, . . . , 16, 17} such that the equation ai −aj = k does not have three distinct solutions for any k > 0.
- 4.設a1,a2,...a8是8個在{1,2,......16,17}集合裡互不相同的整數，證明存在一整數k>0，讓ai-aj=k有最少三個相異解。同時找出7個在{1,2,......16,17}裡互不相同的整數，讓ai-aj=k沒有三個k>0的相異解

代數主題

- 5. Let x, y, and z be non-negative real numbers satisfying $x + y + z = 1$. Show that $(x^2)y + (y^2)z + (z^2)x \leq 4/27$, and find when equality occurs.
- 5.設x,y,z三個非負實數滿足$x+y+z=1$。證明$(x^2)y + (y^2)z + (z^2)x \leq 4/27$,並找出符合等於的情況

- 1.找出所有 $4x^2 - 40[x] + 51 = 0$ 的實數解，如果x是實數，則[x]表示為小於或等於x的最大整數
- $x \geq [x] > x - 1$
- $4x^2 + 51 = 40[x] > 40(x - 1)$

  $4x^2 - 40x + 91 > 0$

  $(2x - 13)(2x - 7) > 0$

  Hence $x > 13/2$ or $x < 7/2$.
- $4x^2 + 51 = 40[x] \leq 40x$

  $4x^2 - 40x + 51 \leq 0$

  $(2x - 17)(2x - 3) \leq 0$

  Hence $3/2 \leq x \leq 17/2$.

- CASE 1: $3/2 \le x < 7/2$.
- For this case, the possible values for $[x]$ are 1, 2 and 3.
- If $[x] = 1$ then $4x^2 + 51 = 40 \cdot 1$ so $4x^2 = -11$, which has no real solutions.
- If $[x] = 2$ then $4x^2 + 51 = 40 \cdot 2$ so $4x^2 = 29$ and $x = \sqrt{29}/2$. Notice that $\sqrt{16}/2 < \sqrt{29}/2 < \sqrt{36}/2$ so $2 < x < 3$ and $[x] = 2$.
- If $[x] = 3$ then $4x^2 + 51 = 40 \cdot 3$ and $x = \sqrt{69}/2$. But $\sqrt{69}/2 > \sqrt{64}/2 = 4$. So, this solution is rejected.

- CASE 2: $13/2 < x \leq 17/2$.
- For this case, the possible values for [x] are 6, 7 and 8.
- If [x] = 6 then $4x^2 + 51 = 40 \cdot 6$ so $x = \sqrt{189}/2$. Notice that $\sqrt{144}/2 < \sqrt{189}/2 < \sqrt{196}/2$ so $6 < x < 7$ and [x] = 6.
- If [x] = 7 then $4x^2 + 51 = 40 \cdot 7$ so $x = \sqrt{229}/2$. Notice that $\sqrt{196}/2 < \sqrt{229}/2 < \sqrt{256}/2$ so $7 < x < 8$ and [x] = 7.
- If [x] = 8 then $4x^2 + 51 = 40 \cdot 8$ so $x = \sqrt{269}/2$. Notice that $\sqrt{256}/2 < \sqrt{269}/2 < \sqrt{324}/2$ so $8 < x < 9$ and [x] = 8.

- The solutions are

$$x = \sqrt{29}/2 , \sqrt{189}/2 , \sqrt{229}/2 , \sqrt{269}/2 .$$

Step 1

1.整理

2.X恒大於等於其高斯符號
其高斯符號又恒大於x-1

3.將 $4x^2+51=40[x]$. 帶入不等式右半邊 $[x]>x-1$

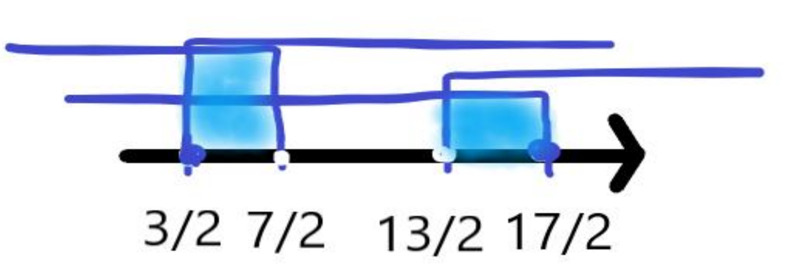
4.從 $(2x-13)(2x-7) > 0$ 得到解 $x>13/2$ or $x<7/2$.

Rearranging the equation we get $4x^2+51=40[x]$. It is known that $x \geq [x] > x-1$, so

$$\begin{aligned} 4x^2+51=40[x] &> 40(x-1) \\ 4x^2-40x+91 &> 0 \\ (2x-13)(2x-7) &> 0 \end{aligned}$$

Step2.

1.解不等式左半邊 $x \geq [x]$
2.帶入 $4x^2 + 51 = 40[x]$
3.得到 $4x^2 + 51 = 40[x] \leq 40x$
4.整理後因式分解得 $(2x - 17)(2x - 3) \leq 0$
5.得解 $3/2 \leq x \leq 17/2.$
6.與Step1的解合併，並得出此圖



$$
\begin{aligned}
4x^2 + 51 = 40[x] &\leq 40x \\
4x^2 - 40x + 51 &\leq 0 \\
(2x - 17)(2x - 3) &\leq 0
\end{aligned}
$$

Hence $3/2 \leq x \leq 17/2$. Combining these inequalities gives $3/2 \leq x < 7/2$ or $13/2 < x \leq 17/2$.