# 數思解第二組報告

## 2013 APMO試題

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# 分析類別

- 問題一:幾何
- 問題二:數論
- 問題三:代數
- 問題四:代數
- 問題五:幾何

**問題一、** 令 $ABC$ 爲一銳角三角形其中 $AD, BE$ 與 $CF$ 爲其高, 且令 $O$ 別爲其外接圓圓心。試證線段 $OA, OF, OB, OD, OC, OE$ 將三角形 $ABC$ 分割爲三對面積相等的三角形。

**Problem 1.** Let $ABC$ be acute triangle with altitudes $AD$, $BE$ and $CF$, and let $O$ be the center of its circumcircle. Show that the segments $OA, OF, OB, OD, OC, OE$ dissect the triangle $ABC$ into three pairs of triangles that have equal areas.

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問題二、試決定所有正整數 $n$ 使得 $\frac{n^2+1}{[\sqrt{n}]^2+2}$ 爲一整數。此處 $[r]$ 表示小於或等於 $r$ 的最大整數。

**Problem 2.** Determine all positive integers $n$ for which $\frac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to $r$.

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**問題三、** 對於 $2k$ 個實數 $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ 定義數列 $X_n$ 如下:

$$X_n = \sum_{i=1}^{k} [a_i n + b_i] \ (n = 1, 2, \cdots).$$

若數列 $X_n$ 形成一等差數列, 試證 $\sum_{i=1}^{k} a_i$ 必爲一整數。此處 $[r]$ 表示小於或等於 $r$ 的最大整數。

**Problem 3.** For $2k$ real numbers $a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_k$ define the sequence of number $X_n$ by

$$X_n = \sum_{i=1}^{k} [a_i n + b_i] \ (n = 1, 2, \cdots).$$

If the sequence $X_n$ forms an aritemetic progression, show that $\sum_{i=1}^{k} a_i$ must be an integer. Here $[r]$ denotes the greatest integer less than or equal to $r$.

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**問題四**、 設 $a, b$ 爲正整數, 且 $A, B$ 是整數中滿足下列兩條件的有限子集:

(i) $A$ 與 $B$ 互斥。

(ii) 若整數 $i$ 屬於 $A$ 或屬於 $B$, 則「$i+a$ 屬於 $A$」與「$i-b$ 屬於 $B$」恰有一成立。

試證: $a\,|A| = b\,|B|$ (這裡 $|X|$ 指的是集合 $X$ 的元素個數)。

**Problem 4.** Let $a$ and $b$ be positive integers, and let $A$ and $B$ be finite sets of integers satisfying:

(i) $A$ and $B$ are disjoint.

(ii) if an integer $i$ belongs either to $A$ or to $B$, then either $i + a$ belongs to $A$ or $i - b$ belongs to $b$.

Prove that $a|A| = b|B|$. (Here $X$ denotes the number of elements in the set $X$.)

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**問題五**、 設四邊形 $ABCD$ 內接於圓 $\omega$, 點 $P$ 位於直線 $AC$ 上, 且直線 $PB$、$PD$ 皆與 $\omega$ 相切。已知過 $C$ 點的圓的切線與直線 $PD, AD$ 分別交於 $Q, R$ 兩點。令 $E$ 點是直線 $AQ$ 與 $\omega$ 的第二個交點。試證: $B, E, R$ 三點共線。

**Problem 5.** Let $ABCD$ be a quadrilateral inscribed in a circle $\omega$, and let $P$ be a point on the extension of $AC$ such that $PB$ and $PD$ are tangent to $\omega$. The tangent at $C$ intersects $PD$ at $Q$ and the line $AD$ at $R$. Let $E$ be the second point of intersection between $AQ$ and $\omega$. Prove that $B, E, R$ are col

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**Problem 2.** Determine all positive integers $n$ for which $\frac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to $r$.

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# 同餘

- **同餘**在數學中是指數論中的一種等價關係
- 當兩個整數除以同一個**正**整數，若得相同餘數，則二整數**同餘**。

兩個整數$a$，$b$，若它們除以正整數$m$所得的餘數相等，則稱$a$，$b$對於模$m$同餘

記作$a \equiv b \pmod{m}$

讀作$a$同餘於$b$模$m$，或讀作$a$與$b$關於模$m$同餘。

比如$26 \equiv 14 \pmod{12}$。

同餘性質之一

k為整數，n為正整數，$(km \pm a)^n \equiv (\pm a)^n \pmod{m}$

# 官方詳解

**Problem 2.** Determine all positive integers $n$ for which $\frac{n^2+1}{[\sqrt{n}]^2+2}$ is an integer. Here $[r]$ denotes the greatest integer less than or equal to $r$.

**Solution.** We will show that there are no positive integers $n$ satisfying the condition of the problem.

Let $m = [\sqrt{n}]$ and $a = n - m^2$. We have $m \geq 1$ since $n \geq 1$. From $n^2+1 = (m^2+a)^2+1 \equiv (a-2)^2+1 \pmod{(m^2+2)}$, it follows that the condition of the problem is equivalent to the fact that $(a-2)^2+1$ is divisible by $m^2+2$. Since we have

$$0 < (a-2)^2+1 \leq \max\{2^2, (2m-2)^2\}+1 \leq 4m^2+1 < 4(m^2+2),$$

we see that $(a-2)^2+1=k(m^2+2)$ must hold with $k=1,2$ or 3. We will show that none of these can occur.

*Case 1.* When $k=1$. We get $(a-2)^2-m^2=1$, and this implies that $a-2=\pm 1$, $m=0$ must hold, but this contradicts with fact $m \geq 1$.

*Case 2.* When $k=2$. We have $(a-2)^2+1=2(m^2+2)$ in this case, but any perfect square is congruent to $0,1,4$ mod 8, and therefore, we have $(a-2)^2+1 \equiv 1,2,5 \pmod 8$, while $2(m^2+2) \equiv 4,6 \pmod 8$. Thus, this case cannot occur either.

*Case 3.* When $k=3$. We have $(a-2)^2+1=3(m^2+2)$ in this case. Since any perfect square is congruent to 0 or 1 mod 3, we have $(a-2)^2+1 \equiv 1,2 \pmod 3$, while $3(m^2+2) \equiv 0 \pmod 3$, which shows that this case cannot occur either.

# 改寫2013 APMO第二題

**Problem 2.** Determine all positive integers n for which $\dfrac{4n^2+1}{[\sqrt{n}\,]^2+2}$

is an integer. Here [r] denotes the greatest integer less then or equal to r.

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Q: $\frac{4n^2+1}{[\sqrt{n}\,]^2+2} \overset{?}{\in} Z^+$

A: $Suppose\ a = [\sqrt{n}\,]$ , b=n $- a^2 \Rightarrow$ n $= a^2 +$ b

$$[\sqrt{n}\,]^2+2 = a^2 + 2$$

$$4n^2 + 1 = 4(a^2 + \mathrm{b})^2+1 = (2a^2 + 2\mathrm{b})^2+1 \equiv (2b-4)^2 + 1\left(mod(a^2+2)\right)$$

$$[\sqrt{n}]^2 \leq n \leq ([\sqrt{n}]^2 + 1)^2 - 1$$

$$a^2 \leq n \leq a^2 + 2\mathrm{a}$$

$$0 \leq n - a^2 \leq 2a$$

$$0 \leq b \leq 2a$$

$$\because 0 \le b \le 2a$$

$$0 < (2b-4)^2+1 \le \mathrm{Max}\{16, \underline{(4a-4)^2\} + 1}$$

$$16a^2 - 32a + 16 + 1 < 16a^2 + 1 < 16a^2 + 32 = 16(a^2+2)$$

$$\frac{4n^2+1}{[\sqrt{n}\,]^2+2} = \frac{(2b-4)^2+1}{a^2+2} < \frac{16(a^2+2)}{a^2+2} = 16$$

$$\text{let } \frac{4n^2+1}{[\sqrt{n}\,]^2+2} = k \qquad k = \{1,2,3,\dots,14,15\}$$

$Case\ 1: when\ k = 1,\ (2b-4)^2+1 = a^2+2$

$$(2b-4)^2 - a^2 = 1$$

$$(2b-4+a)\ (2b-4-a) = 1$$

$$2b-4+a = 1$$

$$+)\ 2b-4-a = -1$$

$$4b-8=0\quad, b=2, a^2=-1, a=\sqrt{-1}\ (\rightarrow\leftarrow)$$

$Case\ 2: when\ k = 2,\ \underbrace{4(b-2)^2+1}_{odd} = \underbrace{2(a^2+2)}_{even}\ (\rightarrow\leftarrow)$

$\therefore k = 2n\ (n \in Z^+)$ Use Case 2. result

$$Case\ 3: when\ k = 3, 4(b-2)^2 + 1 = 3(a^2+2)$$
$$4(b-2)^2+1 \equiv 1, 2\ (mod\ 3)$$
$$3(a^2+2) \equiv 0\ (mod\ 3)$$
$$\therefore k = 3n\ (n \in Z^+)\ \text{Use Case 3. result}$$

$$Case\ 4: when\ k = 5, 4(b-2)^2 + 1 = 5(a^2+2)$$
$$4(b-2)^2 + 1 \equiv 1\ (mod\ 4)$$
$$5(a^2+2) \equiv 2, 3\ (mod\ 4)(\rightarrow\leftarrow)$$

$$Case\ 5: when\ k = 7, 4(b-2)^2 + 1 = 7(a^2+2)$$
$$4(b-2)^2 + 1 \equiv 1,2,3,5 (mod\ 7)$$
$$7(a^2+2) \equiv 0\ (mod\ 7)(\rightarrow\leftarrow)$$

$$Case\ 6: when\ k = 11,\ 4(\mathrm{b}-2)^2+1 = 11(a^2+2)$$

$$4(\mathrm{b}-2)^2+1 \equiv 1,2,4,5,6,10 (mod\ 11)$$

$$11(a^2+2) \equiv \quad 0 \quad (mod\ 11)(\rightarrow\leftarrow)$$

$$Case\ 7: when\ k = 13,\ 4(\mathrm{b}-2)^2+1 = 13(a^2+2)$$

$$4(\mathrm{b}-2)^2+1 \equiv 1 (mod\ 4)$$

$$13(a^2+2) \equiv 5(a^2+2) \equiv 2,3 (mod\ 4)(\rightarrow\leftarrow)$$

$Case\ 1{\sim}7\ shows\ that\ there\ are\ no\ positive\ intergers\ n\ satisfying\ the\ problem.$

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# 參考資料來源

1.維基百科 同餘性質
https://zh.wikipedia.org/wiki/%E5%90%8C%E9%A4%98

2.APMO 歷屆試題
https://www.apmo-official.org/problems

# 謝謝聆聽