2014 APMO

第三組

曾湞湞	林亮辰
410731248	411031107
戴世勳	黃俊穎
411031102	411031113
潘柏銓	李柔樺
411031103	411031123

PROBLEM 1.

QUESTION:

- For a positive integer m **denote by** S(m) and P(m) the sum and **product** ,**respectively**, of the **digits** of m. Show that for each positive integer n, there exist positive integers $a_1, a_2, \ldots, a_n$ satisfying the following **conditions**:

  $$S(a_1) < S(a_2) < \cdots < S(a_n) \text{ and } S(a_i) = P(a_{i+1}) \; (i = 1, 2, \ldots, n).$$

  (We let $a_{n+1} = a_1$.)

  (Problem **Committee** of the Japan Mathematical **Olympiad** Foundation)

單字翻譯:

denote by	表示為	digit	數字	Olympiad	奧林匹克競賽
product	積	condition	狀況		
respectively	分別;個別	committee	委員會		

Question:(中文翻譯)

- 對於正整數 m，分別用 S(m) 和 P(m) 表示數字m的和以及乘積。證明對於每個正整數 n，存在正整數整數 $a_1, a_2, . . . , a_n$，滿足以下條件：

  $S(a_1) < S(a_2) < \cdots < S(a_n)$ 和 $S(a_i) = P(a_{i+1})$ (i = 1, 2, . . . , n).

  (令 $a_{n+1} = a_1$.)

⬇

$S(a_i) = P(a_1)$

Answer:

Solution. Let k be a sufficiently large positive integer. Choose for each $i = 2, 3, \ldots, n$, a_i to be a positive integer among whose digits the number 2 appears exactly $k + i - 2$ times and the number 1 appears exactly $2^{k+i-1} - 2(k + i - 2)$ times, and nothing else. Then, we have $S(a_i) = 2^{k+i-1}$ and $P(a_i) = 2^{k+i-2}$ for each i, $2 \leq i \leq n$. Then, we let a_1 be a positive integer among whose digits the number 2 appears exactly $k + n - 1$ times and the number 1 appears exactly $2^k - 2(k + n - 1)$ times, and nothing else. Then, we see that a_1 satisfies $S(a_1) = 2^k$ and $P(a_1) = 2^{k+n-1}$. Such a choice of a_1 is possible if we take k to be large enough to satisfy $2^k > 2(k + n - 1)$ and we see that the numbers $a_1, \ldots, a_n$ chosen this way satisfy the given requirements.

PROBLEM 2.

- 原文：Problem 2. Let $S = \{1,2,\ldots,2014\}$. For each non-empty subset T ⊆ S, one of its members is chosen as its representative. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset D ⊆ S is a disjoint union of non-empty subsets A, B, C ⊆ S, then the representative of D is also the representative of at least one of A, B, C

翻譯：給定集合$\mathrm{S} = \{1,2,3,,,2014\}$
對任一非空子集$\mathrm{T} \subset \mathrm{S}$，選擇其中一元素為代表
試求指定S所有非空子集代表的方法數，且滿足：
如果一個子集 $D \subseteq S$ 是三個兩兩互斥的非空子集 $A, B, C \subseteq S$ 的聯集,
則 D 的代表必須是 A, B, C 中至少一個的代表

PROBLEM 3.

$Find\ all\ positive\ integers\ n\ such\ \ that\ for\ any\ interger\ k$ $there\ exist\ an\ integer\ a\ for\ which\ \underline{a^3 + a - k\ is\ divisible\ by\ n}.$

(Proposed by Warut Suksompony, Thailand)

試求所有正整數n對任意整數k存在一整數a滿足：$a^3 + a - k$能被n整除。

分析類別：抽象代數　數學結構(群、環、體、模、向量空間)

PROBLEM 4.

QUESTION:

- Let n and b be positive integers. We say n is b-discerning if there exists a set consisting of n different positive integers less than b that has no two different subsets U and V such that the sum of all elements in U equals the sum of all elements in V.
- (a) Prove that 8 is a 100-discerning.
- (b) Prove that 9 is not 100–discerning.

Question:(中文翻譯)

- 設n，b為正整數.令n是一個b-discerning且當存在n個小於b的相異正整數的組合分為兩個相異子集 且兩子集U，V並不相等
- (1)證明8是一個100-discerning
- (2)證明9是一個100-discerning

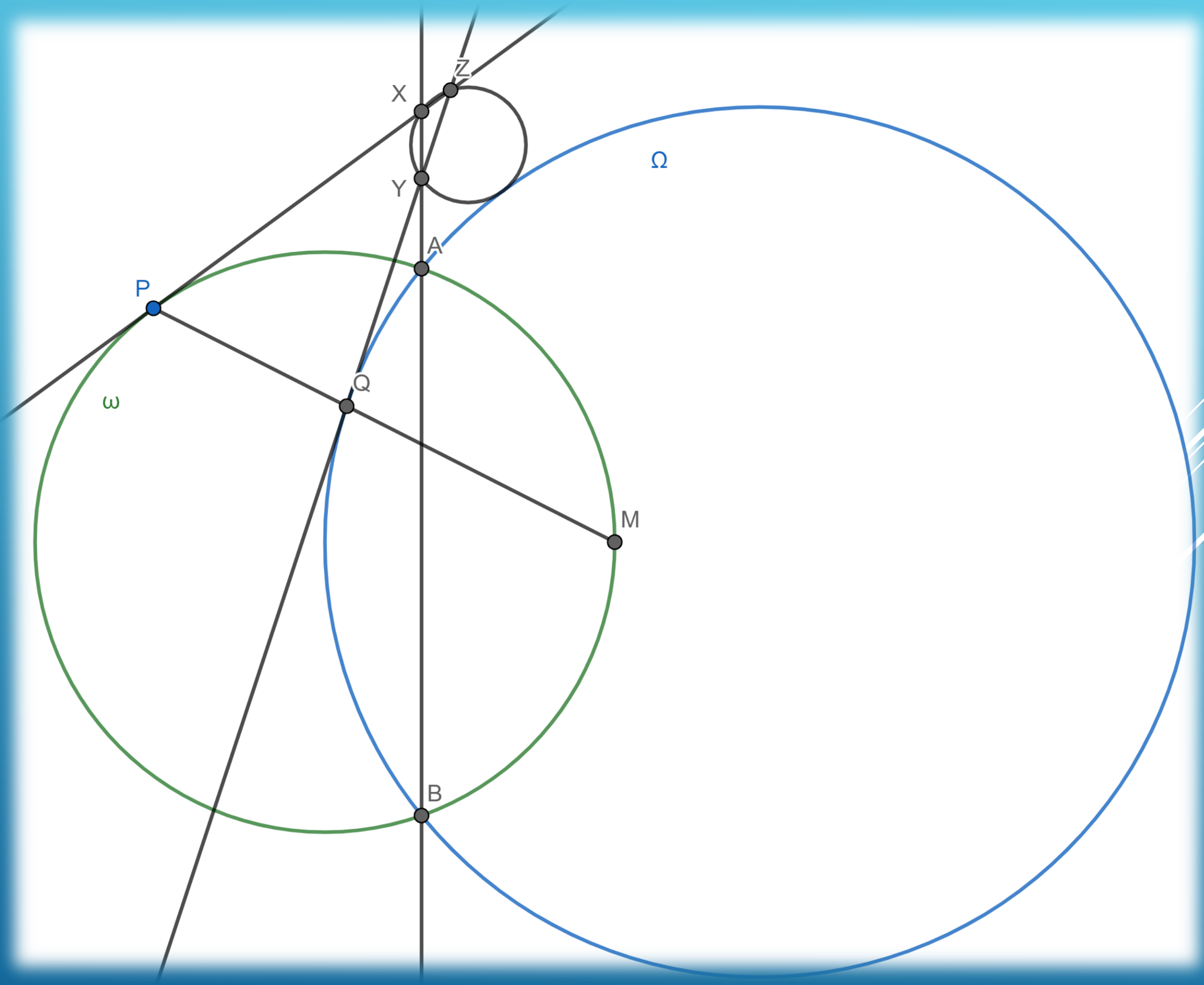
分析類別

- 離散數學

PROBLEM 5.

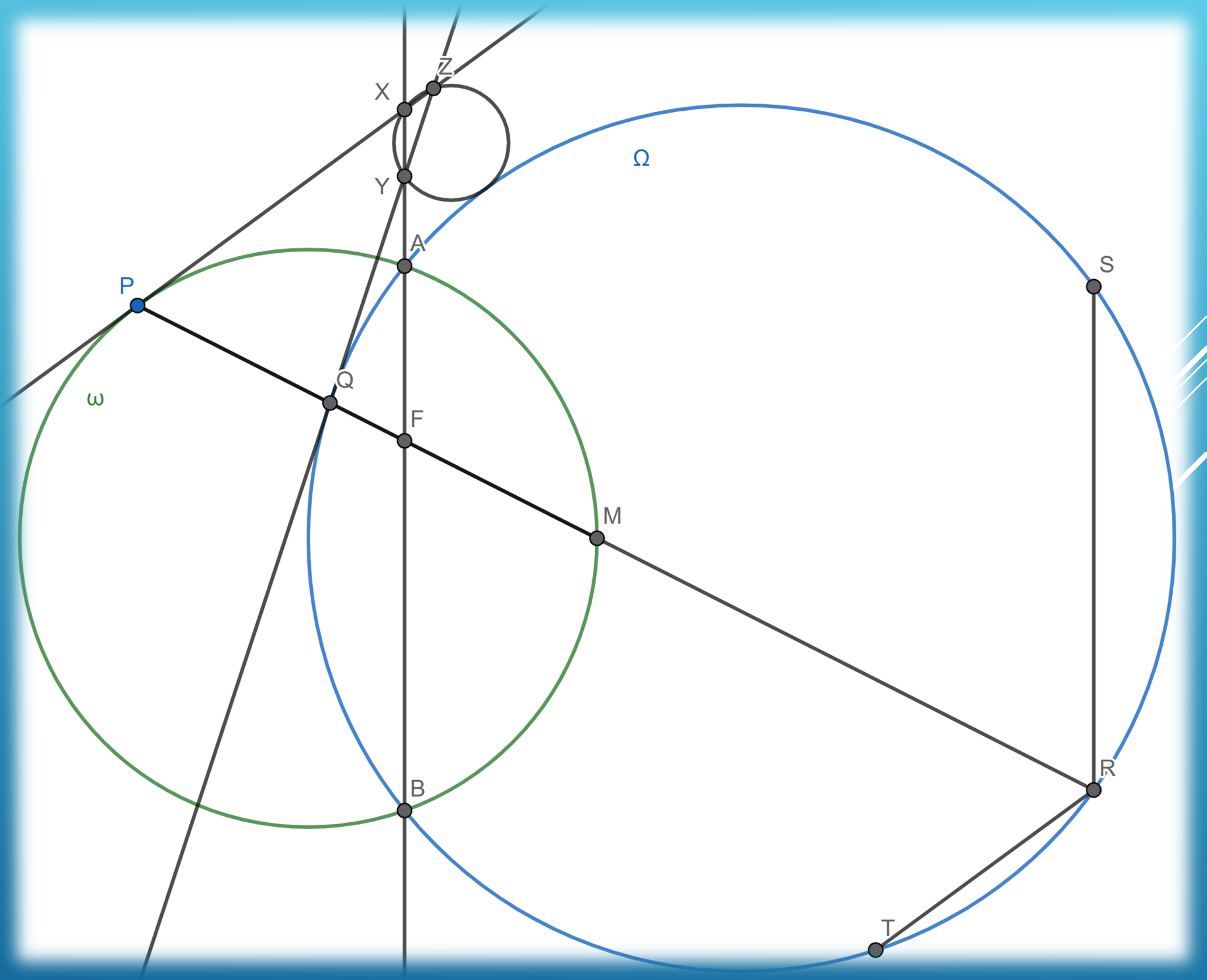
QUESTION:

- Circles ω and Ω meet at points A and B. Let M be the midpoint of the arc AB of circle ω (M lies inside Ω).A chord MP of circle ω intersects Ω at Q (Q lies inside ω).Let l_P be the tangent line to ω at P, and let l_Q be the tangent line to Ω at Q . Prove that the circumcircle of the triangle formed by the lines l_P , l_Q , and AB is tangent to Ω.



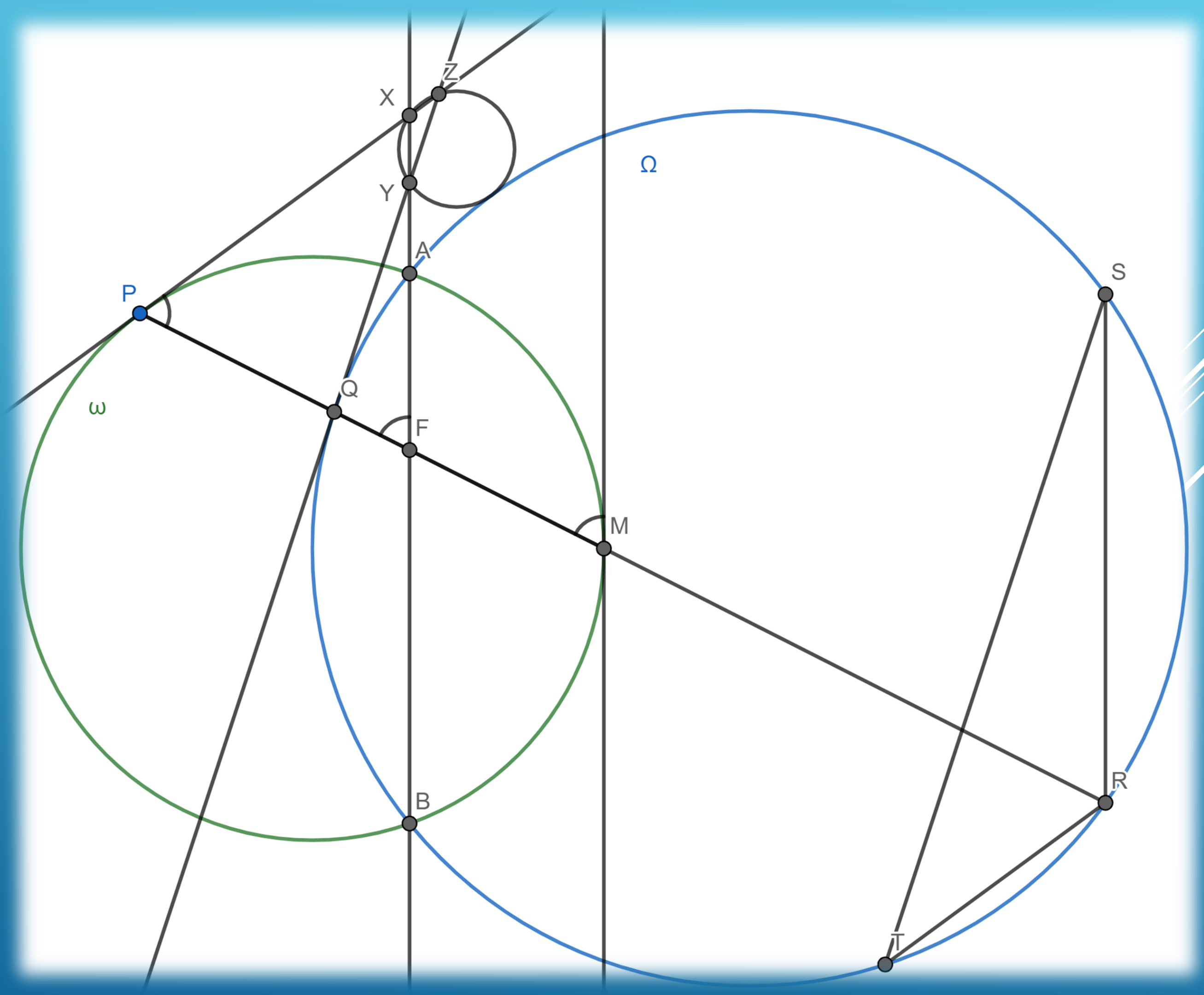
SOLUTION:

- 令F為線段MP跟線段AB的交點
- 找直線PQ與Ω的第二個交點,記為R
- 過R點作與AB線段平行的線,交Ω於S
- 過R點作與P點切線平行的線,交Ω於T



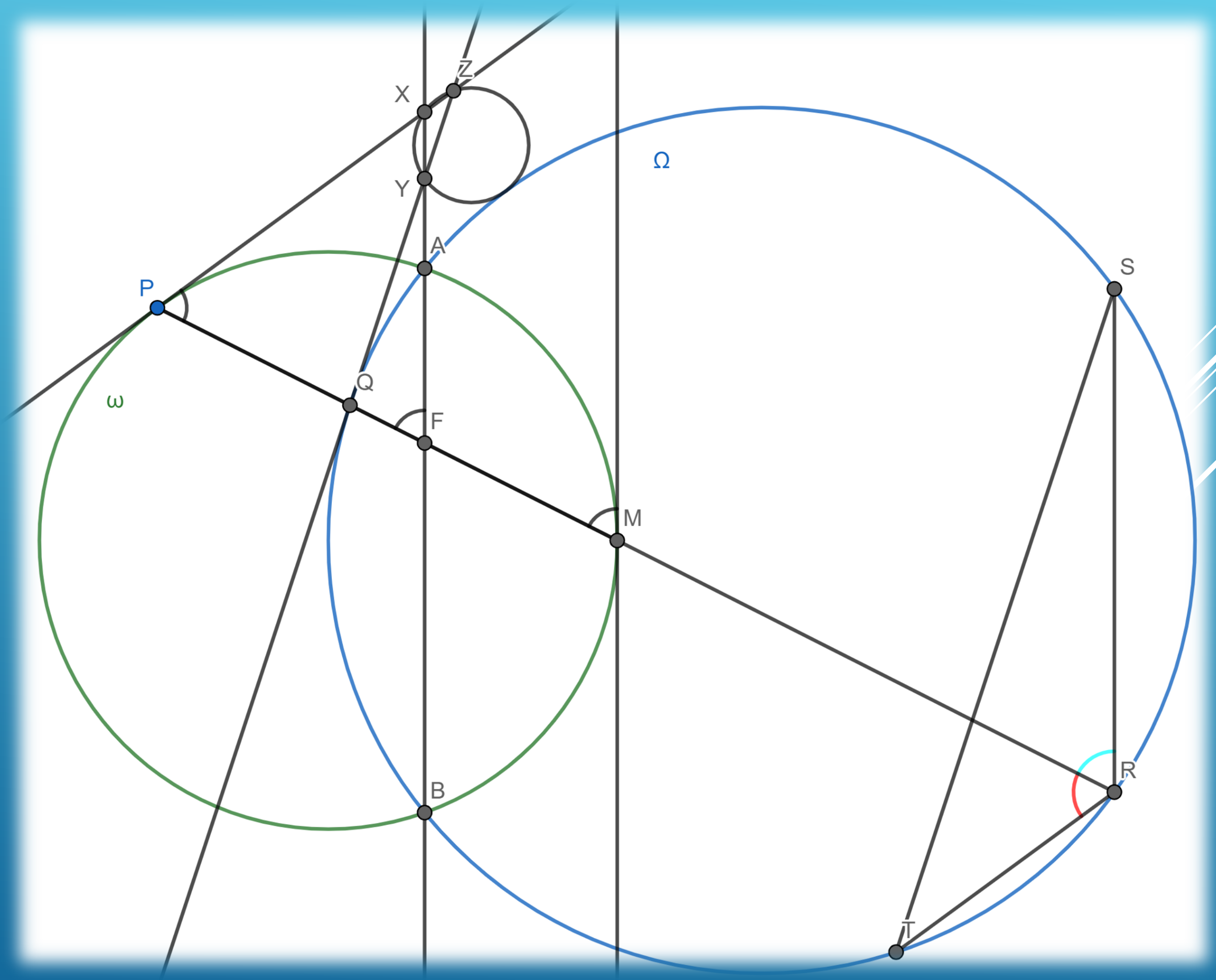
SOLUTION:

- 因為M是弧AB的中點,所以M在ω上的切線與線段AB平行,可得∠M=∠AFP
- 因為∠M及∠P為夾同一弧之弦切角,所以∠M=∠P
- 因此∠M=∠P=∠AFP



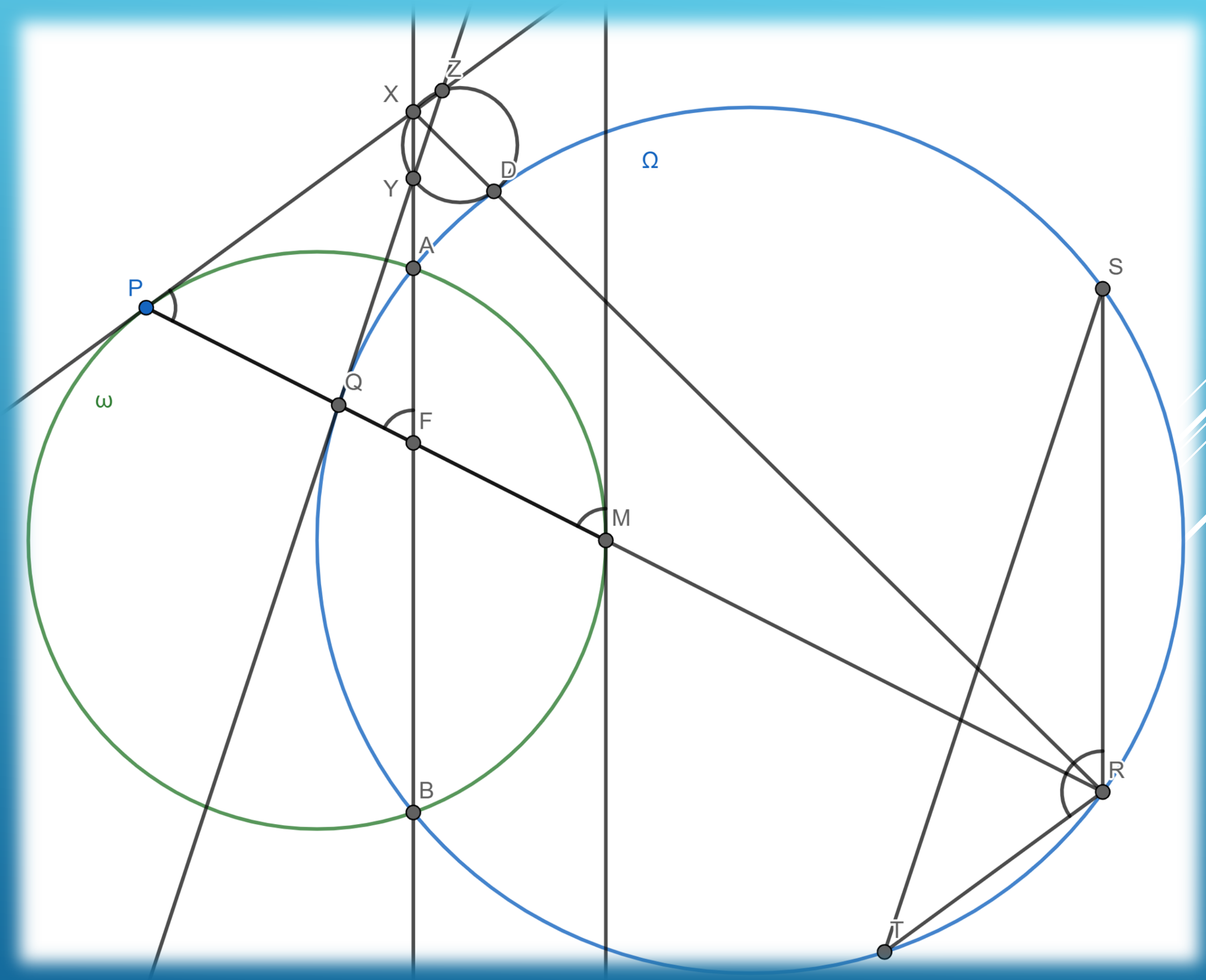
SOLUTION:

- 因為線段RT平行於P的切線,根據內錯角定理,∠PRT=∠P
- 因為線段SR平行於線段AB,根據同位角定理, ∠AFP=∠SRP
- 又∠P=∠AFP,所以∠PRT=∠SRP
- Q為弧TQS的中點(角平分線)
- 點Q的切線與ST線段平行
- 三角形RST與三角形XYZ之對應邊平行



SOLUTION:

- 找直線XR與Ω的第二個交點,記為D
- 宣告D為the center of the homothety h
- 因為D在Ω上,所以三角形XYZ的外接圓與Ω相切
- 須證明D在SY線段上,宣告才成立



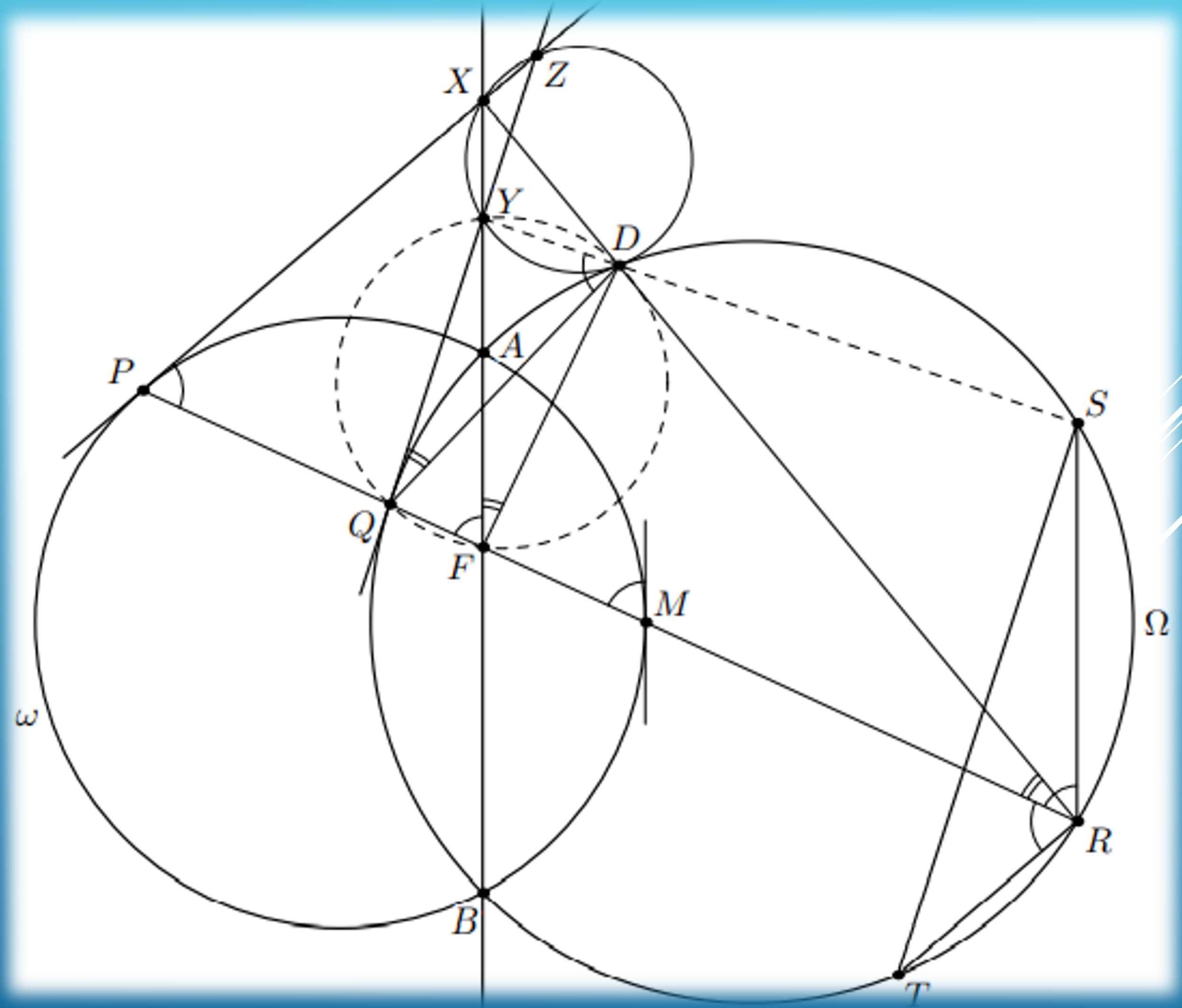
SOLUTION:

- 因為∠P=∠XFP,可得

  $XF^2=XP^2=XA \cdot XB=XD \cdot XR$

  圓切割性質　圓外冪性質

- 所以XF：XD=XR：XF
- 三角形XDF相似於三角形XFR
- ∠DFX=∠XRF=∠DRQ=∠DQY
- D,F,Q,Y共圓　圓內接四邊形
- ∠YDQ=∠YFQ=∠SRQ=180°−∠SDQ
- Y,D,S共線,故得證



類似題

- 圓 Ω 與圓 ω 交於 A, B 兩點。 設圓 ω 上 的 AB 弧之中點為 M (M 位於 Ω 內部)。 圓 ω 上的一弦 MP 與圓 Ω 交於 Q 點 (Q 位於 ω 內部)。 令 l_P 為圓 ω 在 P 點的切線, 而 l_Q 為圓 Ω 在 Q 點的切線。 試問: 由 l_P , l_Q 與 AB 三條直線所形成三角形的外接圓與圓 Ω 是否存在任何關係(相切、相割或無交點)？

組員

曾湞湞
410731248
戴世勳
411031102
潘柏銓
411031103

林亮辰
411031107
黃俊穎
411031113
李柔樺
411031123

~感謝您的聆聽與審閱~