數學思維與解題－作業3(APMO2016)

第五組

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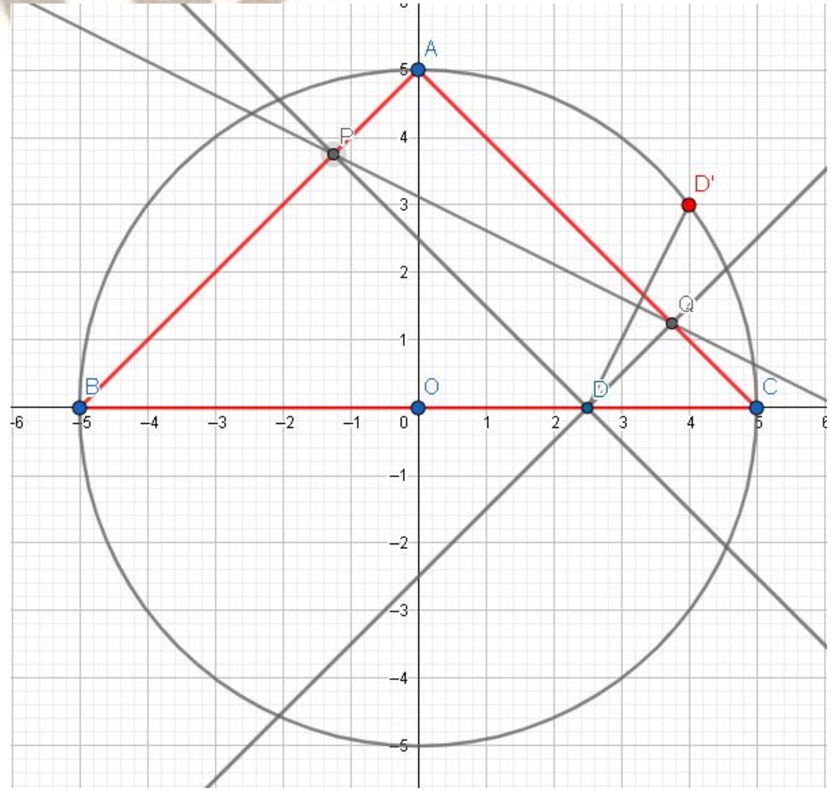
1. We say that a triangle ABC is great if the following holds:

for any point D on the side BC, if P and Q are the feet of the perpendiculars from D to the lines AB, AC, respectively, then the reflection of D in the line P Q lies on the circumcircle of the triangle ABC.

Prove that triangle ABC is great if and only if $\angle A = 90^\circ$ and AB = AC.

翻譯:在以下條件成立的情形下,我們稱三角形 ABC 是〝棒〞的三角形:對 BC 邊上的任意一點 D,若點 P 與 Q 為 D 分別向直線 AB 與 AC 所引垂線的垂足,則 D 對直線 PQ 的鏡射點落在三角形 ABC 的外接圓上。



證明: 三角形 ABC 是〝棒〞的三角形,若且唯若 $\angle A = 90^\circ$ 及 $AB = AC$.

分類:幾何、圓

Solution：

For every point D on the side BC, let D′be the reflection of D in the line PQ.

We will first prove that if the triangle satisfies the condition then it is isosceles(等腰) and right-angled(直角) at A.

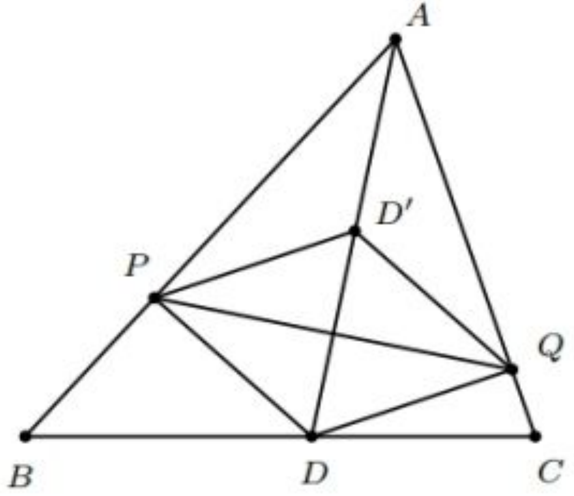
Choose D to be the point where the angle bisector(角平分線) from A meets BC.

Note that P and Q lie on the rays AB and AC respectively.

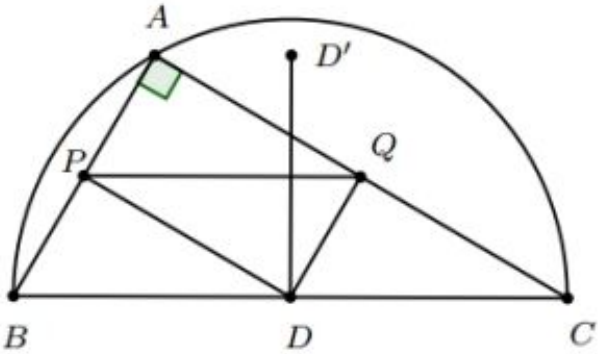
Furthermore, P and Q are reflections of each other in the line AD, from which it follows that PQ ⊥ AD.

Therefore, D′ lies on the line AD and we may deduce that either D′= A or D′ is the second point of the angle bisector at A and the circumcircle(外接圓) of ABC.

However, since APDQ is a cyclic quadrilateral（圓內接四邊形）, the segment PQ intersects the segment AD.
Therefore, D′ lies on the ray DA and therefore D′= A. By angle chasing we obtain $\angle PD'Q = \angle PDQ = 180^\circ - \angle BAC$,and since $D' = A$ we also know $\angle PD'Q = \angle BAC$. This implies that $\angle BAC = 90^\circ$.



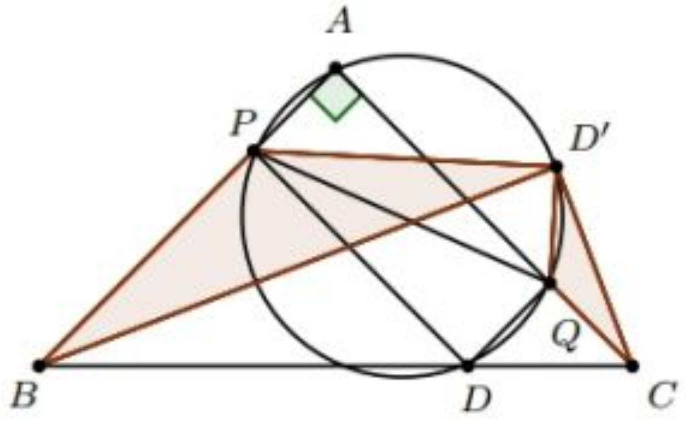
Now we choose D to be the midpoint of BC. Since $\angle BAC = 90^\circ$, we can deduce that DQP is the medial triangle of triangle ABC. Therefore, $PQ \| BC$ from which it follows that $DD' \perp BC$. But the distance from D′ to BC is equal to both the
circumradius of triangle ABC and to the distance from A to BC. This can only happen if $A = D'$. This implies that ABC is isosceles and right-angled at A.



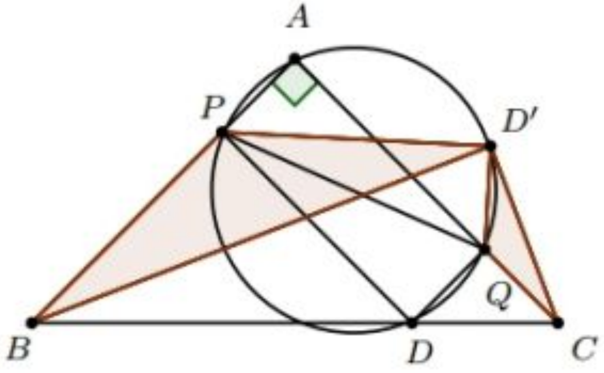
We will now prove that if ABC is isosceles and right-angled at A then the required property in the problem holds.

Let D be any point on side BC. Then D′P = DP and we also have DP = BP. Hence, D′P = BP and similarly D′Q = CQ.

Note that AP DQD′ is cyclic with diameter PQ(直徑).

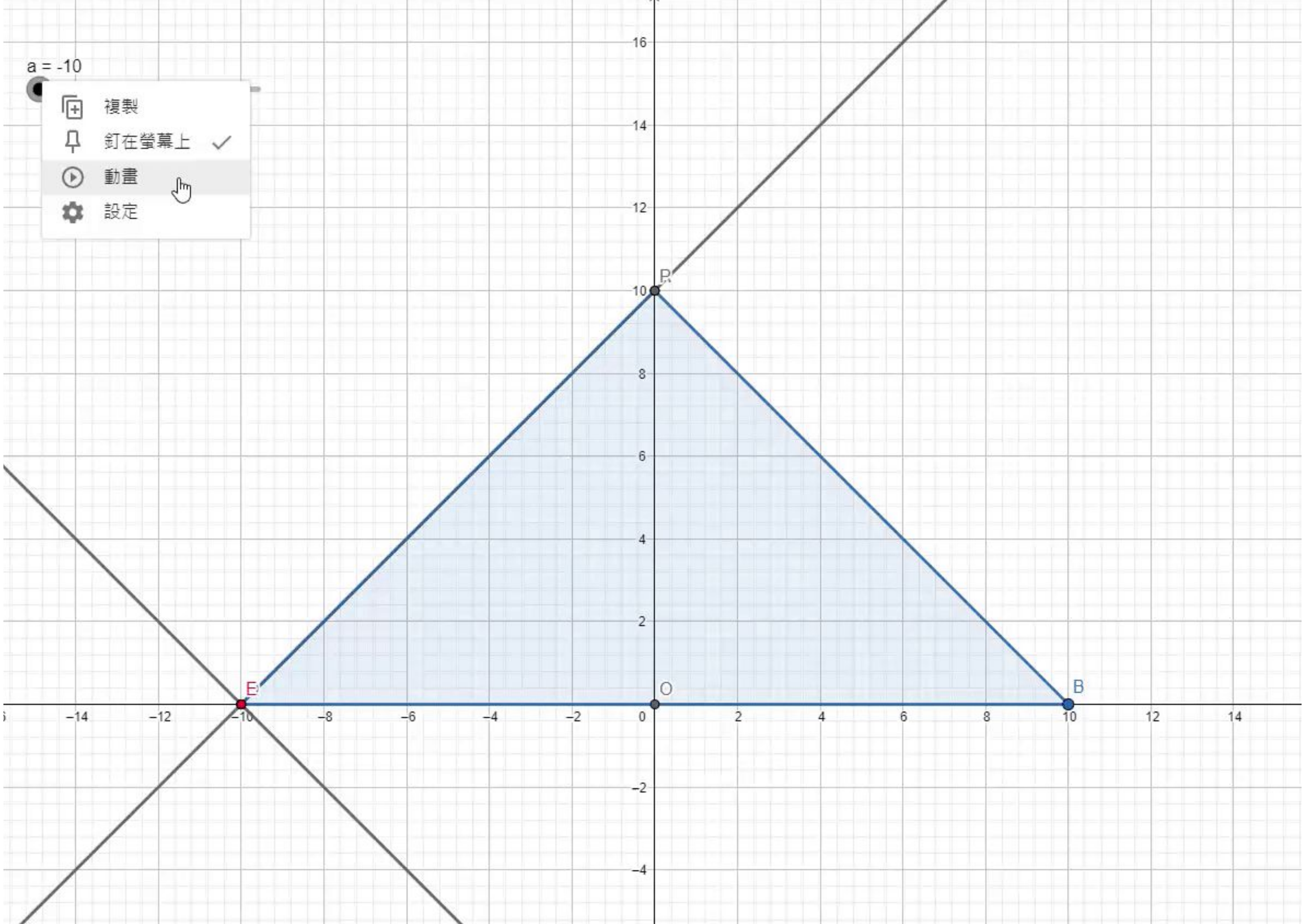


Therefore, $\angle APD' = \angle AQD'$, from which we obtain $\angle BPD' = \angle CQD'$. So triangles D′PB and D′QC are similar.
It follows that $\angle PD'Q = \angle PD'C - \angle CD'Q = \angle PD'C - \angle BD'P = \angle BD'C$ and $D'P / D'Q = D'B / D'C$.
So we also obtain that triangles D′P Q and D′BC are similar. But since DPQ and D′PQ are congruent(全等), we may deduce that $\angle BD'C = \angle PD'Q = \angle PDQ = 90°$. Therefore, D′ lies on the circle with diameter(直徑) BC, which is the circumcircle of triangle ABC.



相似題

三角形 ABC, ∠A = 90° 及 AB = AC, D 為 BC 邊上的任意一點, 若點 P 與 Q 分別為 D 對直線 AB 與 AC 所做垂線的垂足, 設D′為 D 對直線 PQ 的鏡射點, 則所有 D′所形成的軌跡的圖形為何？


a = -10
複製
釘在螢幕上
動畫
設定
P
O
E
B
16
14
12
10
8
6
4
2
0
−2
−4
−14
−12
−10
−8
−6
−4
−2
2
4
6
8
10
12
14

2.A positive integer is called fancy if it can be expressed in the form

$$2^{a1}+2^{a2}+\cdots+2^{a100}$$

, where a1, a2, · · · , a100 are non-negative integers that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number.

翻譯:一個正整數被稱為〝花俏數”, 如果它可以被表示成

$$2^{a1}+2^{a2}+\cdots+2^{a100}$$

, 其中 a1, a2, · · · , a100 為非負整數; 它們不需兩兩相異。試求最小的正整數 n, 使得 n 的任意倍數都不是花俏數。

分類:代數、整數

3.Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects line AB at P. Let N be the intersection of lines P R and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

翻譯:設 AB 與 AC 是不在同一條直線上的兩條射線, 並設圓 ω 的圓心為 O, 且與射線 AC 切於點 E 與射線 AB 切於點 F. 設 R 為線段 EF 上的一點。設過 O 點和 EF 平行的直線, 交直線 AB 於點 P. 令點 N 為直線 P R 及 AC 的交點, 並令點 M 為直線 AB 及過 R 且平行於 AC 的直線的交點。證明: 直線 MN 與圓 ω 相切。

分類:幾何、圓

4.The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly on flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

翻譯:夢想國共有 2016 個城市。星航航空公司想在其中若干對城市中建立單向 的航線, 使得每個城市都恰有一條航線從該城市向外飛出。求最小的正整數 k, 滿足: 不論星航如何規畫航線, 我們總可以將所有城市分成 k 組, 使得任何城市都無 法在 28 段航線內, 抵達同組的另一城市。

分類:組合

5.Find all functions $f : R+ \to R+$ such that

$$(z + 1)f(x + y) = f(xf(z) + y) + f(yf(z) + x)$$

, for all positive real numbers x, y, z

翻譯:試求所有的函數 $f : R+ \to R+$ 使得

$$(z + 1)f(x + y) = f(xf(z) + y) + f(yf(z) + x)$$

, 對所有的正實數 x, y, z

分類:函數

感謝聆聽