第四組

柯西不等式延伸與推廣

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柯西不等式

H¨older 不等式

Young’s 不等式

1

柯西不等式

1 柯西不等式

◆向量表示法：$|\vec{a}\cdot\vec{b}|\leq|\vec{a}|\cdot|\vec{b}|$，等號成立時，$\vec{a}\parallel\vec{b}$。

◆設 a_1，a_2，a_3，b_1，b_2，$b_3\in R$，則 $(a_1^2+a_2^2)(b_1^2+b_2^2)\geq(a_1b_1+a_2b_2)^2$，等號成立時，$\dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}$。

◆同理可推廣至空間向量，亦即 $(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)\geq(a_1b_1+a_2b_2+a_3b_3)^2$，等號成立時，$\dfrac{a_1}{b_1}=\dfrac{a_2}{b_2}=\dfrac{a_3}{b_3}$。

2 柯西不等式的問題

a b c為正整數，且a+b+c=1，求
(a+1/a)^2 + (b+1/b)^2 + (c+1/c)^2之最小值

sol: $\left[\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2\right]\left[1^2+1^2+1^2\right] \geq \left(a+\frac{1}{a}+b+\frac{1}{b}+c+\frac{1}{c}\right)^2$ —①

$= \left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2$

$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c) \geq (1+1+1)^2 = 9$ —②

by ① ② $\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2 \geq \frac{1}{3}(1+9)^2 = \frac{100}{3}$ #

3 柯西不等式問題的額外討論

討論①,②之間連接.

在①的等號成立時 $a+\frac{1}{a}=b+\frac{1}{b}=c+\frac{1}{c}$

在②的等號成立 $\frac{\frac{1}{a}}{a}=\frac{\frac{1}{b}}{b}=\frac{\frac{1}{c}}{c} \Rightarrow a=b=c$

⇒表示等號成立 #

2

H¨older 不等式
Young’ s 不等式

4 hölder 不等式

設 $x_i, y_i \in \mathbb{R}$ $i=1,2,\cdots,n$

則 $(|x_1|^p+|x_2|^p+\cdots+|x_n|^p)^{\frac{1}{p}}+(|y_1|^q+|y_2|^q+\cdots+|y_n|^q)^{\frac{1}{q}}$

$\geq |x_1y_1|+|x_2y_2|+\cdots+|x_ny_n|$

$$\left(\left(\sum_{i=1}^{n}|x_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{n}|y_i|^q\right)^{\frac{1}{q}} \geq \sum_{i=1}^{n}|x_iy_i|\right)$$

其中 $\frac{1}{p}+\frac{1}{q}=1$, $p,q>1$

5 Young’s inquality(證明需要)

設 a, b, p, q 正實數，且 $\frac{1}{p}+\frac{1}{q}=1$

則 $ab \le \frac{a^p}{p} + \frac{b^q}{q}$

等號成立 $\Leftrightarrow a^p = b^q$

$(\because$ 此時，$ab = a(b^q)^{\frac{1}{q}} = a \cdot a^{\frac{p}{q}} = a^p = \frac{a^p}{p} + \frac{b^q}{q})$

6 hölder 不等式的證明

- By Young's inequality for

$$a=\frac{x_i}{\left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}}\quad , \quad b=\frac{y_i}{\left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}}\qquad i=1,2\cdots,n$$

$$\Rightarrow \frac{x_i y_i}{\left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}} \le \frac{1}{p}\frac{x_i^p}{\left(\sum_{i=1}^{n} x_i^p\right)} + \frac{1}{q}\frac{y_i^q}{\left(\sum_{i=1}^{n} y_i^q\right)}$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i y_i}{\left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}} \le \frac{1}{p}\frac{\sum_{i=1}^{n} x_i^p}{\sum_{i=1}^{n} x_i^p} + \frac{1}{q}\frac{\sum_{i=1}^{n} y_i^q}{\left(\sum_{i=1}^{n} y_i^q\right)} = \frac{1}{p}+\frac{1}{q}=1$$

$$\Rightarrow \sum_{i=1}^{n} x_i y_i \le \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}$$

7 hölder 不等式的其他寫法

已知 $f, g \in C_1[a,b]$，$\frac{1}{p}+\frac{1}{q}=1$ 且 $p, q \geq 1$

則 $\left|\int_a^b f(x)g(x)\,dx\right| \leq \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\,dx\right)^{\frac{1}{q}}$

or

$f_1 \cdots, f_n \in C[a,b]$ 且 $\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_n}=1$ $p_i \geq 1$

則 $\left|\int_a^b f_1(x) f_2(x) \cdots f_n(x)\,dx\right| \leq \left(\int_a^b |f_1(x)|^{p_1}\,dx\right)^{\frac{1}{p_1}} \cdots \left(\int_a^b |f_n(x)|^{p_n}\right)^{\frac{1}{p_n}}$

8 hölder 不等式的問題

設 $0<\theta<\frac{\pi}{2}$，$\frac{2}{\sin\theta}+\frac{3}{\cos\theta}$ 最小值為何？

$$\left[(\sin^{\frac{2}{p}}\theta)^{p}+(\cos^{\frac{2}{p}}\theta)^{p}\right]^{\frac{1}{p}}\times\left[\left(\frac{2^{\frac{1}{q}}}{\sin^{\frac{1}{q}}\theta}\right)^{q}+\left(\frac{3^{\frac{1}{q}}}{\cos^{\frac{1}{q}}\theta}\right)^{q}\right]^{\frac{1}{q}}$$

$$\geq \sin^{\frac{2}{p}}\theta\cdot\frac{2^{\frac{1}{q}}}{\sin^{\frac{1}{q}}\theta}+\cos^{\frac{2}{p}}\theta\cdot\frac{3^{\frac{1}{q}}}{\cos^{\frac{1}{q}}\theta}$$

$$\Rightarrow\begin{cases}\frac{1}{p}+\frac{1}{q}=1\\ \frac{2}{p}=\frac{1}{q}\end{cases}\Rightarrow\begin{cases}\frac{1}{p}=\frac{1}{3}\\ \frac{1}{q}=\frac{2}{3}\end{cases}$$

$$\Rightarrow\left[(\sin^{\frac{2}{3}}\theta)^{3}+(\cos^{\frac{2}{3}}\theta)^{3}\right]^{\frac{1}{3}}\geq \sin^{\frac{2}{p}}\theta\cdot\frac{2^{\frac{1}{q}}}{\sin^{\frac{1}{q}}\theta}+\cos^{\frac{2}{p}}\theta\cdot\frac{3^{\frac{1}{q}}}{\cos^{\frac{1}{q}}\theta}\geq 2^{\frac{2}{3}}+3^{\frac{2}{3}}$$

$$\Rightarrow\frac{2}{\sin\theta}+\frac{3}{\cos}\geq\left(2^{\frac{2}{3}}+3^{\frac{2}{3}}\right)^{\frac{3}{2}}\ \#$$

9 Minkowski inequality

設 $x_i, y_i \geq 0 \quad i=1,2,\cdots,n$

則 $(x_1^p + x_2^p + \cdots + x_n^p)^{\frac{1}{p}} + (y_1^p + y_2^p + \cdots + y_n^p)^{\frac{1}{p}} \geq \left[(x_1+y_1)^p + \cdots + (x_n+y_n)^p\right]^{\frac{1}{p}}$

$p>1$

等號成立 $\Leftrightarrow x_i = k y_i \quad 1 \leq i \leq n, k \geq 0$ or $y_i = 0$

10 Minkowski證明(利用Hölder)

$$\sum_{i=1}^{n}(x_i+y_i)^p = \sum_{i=1}^{n}(x_i+y_i)(x_i+y_i)^{p-1}$$

$$= \sum_{i=1}^{n}x_i(x_i+y_i)^{p-1} + \sum_{i=1}^{n}y_i(x_i+y_i)^{p-1}$$

$$\le \left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}\left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{p-1}{p}} + \left(\sum_{i=1}^{n}y_i^p\right)^{\frac{1}{p}}\left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{p-1}{p}}$$

$$= \left[\left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}y_i^p\right)^{\frac{1}{p}}\right]\left[\sum_{i=1}^{n}(x_i+y_i)^p\right]^{\frac{p-1}{p}}$$

$$\Rightarrow \left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}y_i^p\right)^{\frac{1}{p}}$$

11 發現

1 在p=2時會看到我們熟悉的三角不等式，我們可以知道三角不等式其實就是Minkowski不等式的特例。

2 在建造N維空間中，Minkowski不等式佔有重要的一席之地，因為在建構空間時，也要符合三角不等式

12 問題

設 a.b.c 為三角形的邊長，且 $a+b+c=1$

令 $n>1$, $n\in\mathbb{Z}$ ，試證

$$\sqrt[n]{a^n+b^n}+\sqrt[n]{b^n+c^n}+\sqrt[n]{c^n+a^n}<1+\frac{\sqrt[n]{2}}{2}$$

13 解答

設 $x=\frac{-a+b+c}{2}>0$, $y=\frac{a-b+c}{2}>0$, $z=\frac{a+b-c}{2}>0$

by Minkowski 不等式.

$$\sqrt[n]{(y+z)^n+(x+z)^n} \le \sqrt[n]{y^n+x^n}+\sqrt[n]{z^n+z^n}$$

$$\sqrt[n]{(z+x)^n+(y+x)^n} \le \sqrt[n]{z^n+y^n}+\sqrt[n]{x^n+x^n}$$

$$\sqrt[n]{(x+y)^n+(y+z)^n} \le \sqrt[n]{x^n+z^n}+\sqrt[n]{y^n+y^n}$$

$$\Rightarrow \sqrt[n]{a^n+b^n}+\sqrt[n]{b^n+c^n}+\sqrt[n]{c^n+a^n} \le \sqrt[n]{x^n+y^n}+\sqrt[n]{y^n+z^n}+\sqrt[n]{z^n+x^n} + \frac{\sqrt[n]{2}}{2}$$

by 二項定理 $\left(x^n+y^n<(x+y)^n \Rightarrow \sqrt[n]{x^n+y^n}<(x+y)\right)$

$$\therefore \sqrt[n]{a^n+b^n}+\sqrt[n]{b^n+c^n}+\sqrt[n]{c^n+a^n} < (x+y)+(y+z)+(z+x)+\frac{\sqrt[n]{2}}{2}$$
$$= 2(x+y+z)+\frac{\sqrt[n]{2}}{2}$$
$$= 1+\frac{\sqrt[n]{2}}{2}$$

14 Prove the following inequalities

Holder inequality: $\sum_{j=1}^{\infty}|\xi_j\eta_j| \le \left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{\frac{1}{p}}\left(\sum_{m=1}^{\infty}|\eta_m|^q\right)^{\frac{1}{q}},$

where $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1.$

Cauchy-Schwarz inequality: $\sum_{j=1}^{\infty}|\xi_j\eta_j| \le \left(\sum_{k=1}^{\infty}|\xi_k|^2\right)^{\frac{1}{2}}\left(\sum_{m=1}^{\infty}|\eta_m|^2\right)^{\frac{1}{2}}.$

Minkowski inequality: $\left(\sum_{j=1}^{\infty}|\xi_j+\eta_j|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^{\infty}|\eta_m|^p\right)^{\frac{1}{p}},$

where $p > 1.$

15 參考連結

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