第六組

Senior Fall, A-Level 2011

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幾何

1. Pete has marked several (three or more) points in the plane such that all distances between them are different. A pair of marked points A; B will be called unusual if A is the furthest marked point from B, and B is the nearest marked point to A (apart from A itself). What is the largest possible number of unusual pairs that Pete can obtain?

1. 小皮在平面上標記了至少三個點，使得任兩點之間的距離都互不相同。若從點 B 出發到其它的點之距離最長的點是 A；從點 A 出發到其它的點之距離最短的點是 B，則稱點 A 與點 B 是一對「奇特的點對」。請問小皮最多可以得出幾對「奇特的點對」？

3.In triangle ABC, points A_1, B_1, $C_1$are bases of altitudes from vertices A, B, C, and points C_A, $C_B$are the projections of C_1 to AC and BC respectively. Prove that line C_AC_B bisects the segments $C_1A_1$and C_1B_1.

3.在△ABC 中，點 A_1, B_1, C_1分別為從點 A、B、C 引出之高的垂足。點 C_A, C_B為點 C_1分別對 AC、BC 的投影。請證明直線 C_AC_B平分線段 C_1A_1與 C_1B_1。

4.Does there exist a convex N-gon such that all its sides are equal and all vertices belong to the parabola $y = x^2$for

a) N = 2011;   b) N = 2012?

4.是否存在一個凸 n 邊形使得它的頂點都在拋物線 $y = x^2$ 上且它的邊長都相 等？

(a) 當 n=2011 時； (b) 當 n=2012 時？

幾何&代數

7.100 red points divide a blue circle into 100 arcs such that their lengths are all positive integers from 1 to 100 in an arbitrary order. Prove that there exist two perpendicular chords with red endpoints.

7.在一個藍色的圓周上有 100 個紅點，這些紅點把圓周分割為 100 段弧，使得這 100 段弧之弧長恰好為 1、2、3、...、100 單位以某種順序排列。請證明必定存在兩條互相垂直的弦且它們的端點都是紅點。

代數

5.We will call a positive integer good if all its digits are nonzero. A good integer will be called special if it has at least k digits and their values strictly increase from left to right. Let a good integer be given. At each move, one may either add some special integer to its digital expression from the left or from the right, or insert a special integer between any two its digits, or remove a special number from its digital expression. What is the largest k such that any good integer can be turned into any other good integer by such moves?

5.若一個正整數的每位數碼都不為 0，則我們稱此數為一個好數。若一個好數的數碼由左至右嚴格遞增，則稱此數為一個「奇異數」。現給定一個好數，每次操作可以在此數的數碼之左側、右側或中間添入或移除一個至少有 k 位數的「奇異數」。請問能將任何一個好數經過有限次上述操作變成任意另一個好數的最大之 k 值是什麼？

6. Prove that the integer $1^1 + 3^3 + 5^5 + \cdots + (2n-1)^{2^n-1}$ is a multiple of 2^n but not a multiple of 2^{n+1} .

6.請證明 $1^1 + 3^3 + 5^5 + \cdots + (2n-1)^{2^n-1}$之值可以被$2^n$整除，但是不可以被 2^{n+1}整除。

2.Given that $0 < a, \mathrm{b}, \mathrm{c}, \mathrm{d} < 1$and

$$\mathrm{abcd} = (1-a)(1-b)(1-c)(1-d),$$

prove that $(\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}) - (\mathrm{a}+\mathrm{c})(\mathrm{b}+\mathrm{d}) \geq 1.$

2.已知$0 < a, \mathrm{b}, \mathrm{c}, \mathrm{d} < 1$且$\mathrm{abcd} = (1-a)(1-b)(1-c)(1-d)$。

請證明：

$$(\mathrm{a}+\mathrm{b}+\mathrm{c}+\mathrm{d}) - (\mathrm{a}+\mathrm{c})(\mathrm{b}+\mathrm{d}) \geq 1$$

【參考解法 1】

由 $abcd=(1-a)(1-b)(1-c)(1-d)$ 可得：

$$
\begin{aligned}
& a+b+c+d-(a+c)(b+d) \\
&=1+ac(1-b-d)+bd(1-a-c) \\
&=1+ac(1-b)(1-d)+bd(1-a)(1-c)-2abcd \\
&\geq 1+2\sqrt{ac(1-b)(1-d)bd(1-a)(1-c)}-2abcd \\
&=1+2abcd-2abcd \\
&=1
\end{aligned}
$$

【參考解法 2】

由 $abcd=(1-a)(1-b)(1-c)(1-d)$ 可得 $\dfrac{ac}{(1-a)(1-c)}=\dfrac{(1-b)(1-d)}{bd}$，故知

$$\frac{a+c-1}{(1-a)(1-c)}=\frac{ac}{(1-a)(1-c)}-1=\frac{(1-b)(1-d)}{bd}-1=\frac{1-b-d}{bd}\text{。}$$

再由 $\dfrac{a+c-1}{(1-a)(1-c)}=\dfrac{1-b-d}{bd}$ 可知 $\dfrac{(a+c-1)(1-b-d)}{(1-a)(1-c)bd}\geq 0$ 。

因 $(1-a)(1-c)bd>0$，故可得 $(a+c-1)(1-b-d)\geq 0$，此即

$$a-ab-ad+c-cb-cd-1+b+d\geq 0$$

$$(a+b+c+d)-(a+c)(b+d)\geq 1$$

延伸

given that $0<x,y,z,w<1$

and

$1=(1/(x^{2022})-1)(1/(y^{2022})-1)(1/(z^{2022})-1)(1/(w^{2022})-1)$

prove that

$x^{2022}+y^{2022}+z^{2022}+w^{2022}-(x^{2022}+y^{2022})(z^{2022}+w^{2022})>=1$

解法

0<x,y,z,w<1 -> 0<x^2022,y^2022,z^2022,w^2022<1

令a=x^2022,b=y^2022,c=z^2022,d=w^2022

由 $abcd=(1-a)(1-b)(1-c)(1-d)$ 可得：

$$
\begin{aligned}
& a+b+c+d-(a+c)(b+d) \\
&= 1+ac(1-b-d)+bd(1-a-c) \\
&= 1+ac(1-b)(1-d)+bd(1-a)(1-c)-2abcd \\
&\geq 1+2\sqrt{ac(1-b)(1-d)bd(1-a)(1-c)}-2abcd \\
&= 1+2abcd-2abcd \\
&= 1
\end{aligned}
$$


THANK YOU