數學思維與解題－作業1
第九組

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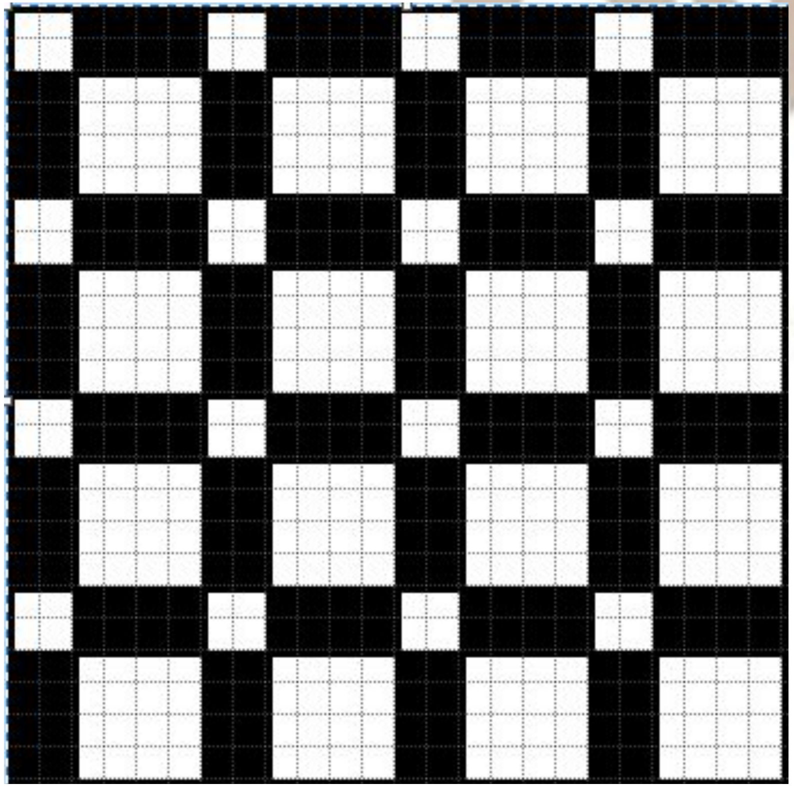
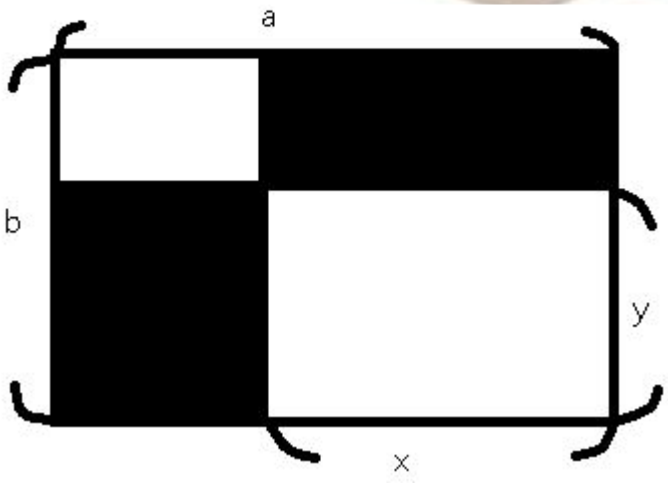
1. A square board is divided by lines parallel to the board sides (7 lines in each direction, not necessarily equidistant) into 64 rectangles. Rectangles are colored into white and black in alternating order. Assume that for any pair of white and black rectangles the ratio between area of white rectangle and area of black rectangle does not exceed 2. Determine the maximal ratio between area of white and black part of the board. White (black) part of the board is the total sum of area of all white (black) rectangles.

翻譯:一個正方形板, 被平行線分割成64塊長方形, 直橫都有7條線, 每塊面積不必相等, 將長方形黑白相間塗色, 若任意一個白色長方形的面積最多是任意一個黑色格子面積的 2 倍, 請問所有白色格子總面積至多是所有黑色格子總面積的多少倍？

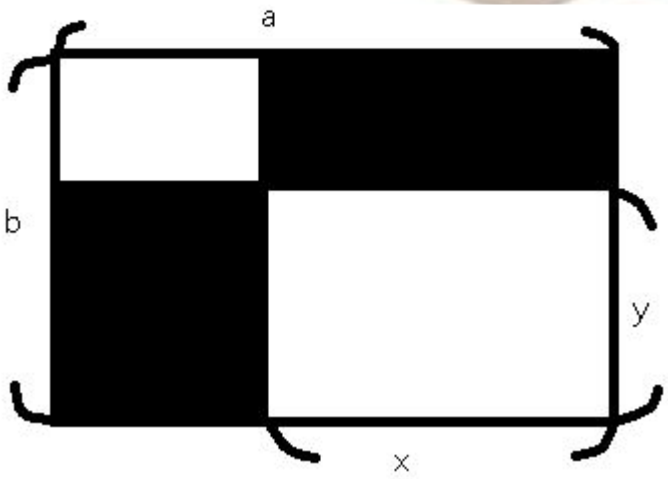


We may have rows and columns of alternating widths 1/3 and 2/3 . Let the white cells remain squares while the black cells become non-squares. Then the area of each white cell is either 1/9 or 4/9 while the area of each black cell is 2/9 .

Thus the ratio between the area of any white cell and the area of any black cell is at most 2. The total area of the white cells is 16(1/9 + 4/9) = 80/9 while the total area of the black cells is 32(2/9) = 64/9 . Here, the ratio of the total area of the white cells to the total area of the black cells is 80/9 : 64/9 = 5 : 4.



To show that this is the maximum possible, divide the modified chessboard into 16 subboards each consisting of four cells in a 2×2 configuration. Let the dimensions of one of the subboards be $a \times b$. Let the vertical grid line divide it into two rectangles of widths x and $(a - x)$, and we may assume that $x > a/2$. Let the horizontal grid line divide the subboard into two rectangles of heights y and $(b - y)$, and we may assume that $y > b/2$. The condition that the ratio between the area of any white cell and the area of any black cell is at most 2 applies here also, and this is satisfied if and only if $x \leq 2a/3$ and $y \leq 2b/3$.



Let the white cells be $(x \times y)$ and $(a - x)(b - y)$.

Area : $T = 2xy + ab - bx - ay = x(2y - b) + a(b - y)$.

$= y(2x - a) + b(a - x)$.

Since $2y > b$, T increases as x increases to its maximum value of $2a/3$; Since $2x > a$, T increases as y increases to its maximum value of $2b/3$.

Hence $T \leq 8ab/9 + ab - 2(2ab/3) = (5/9)ab$.

It follows the ratio of the total area of the white cells to the total area of the black cells is at most 5:4. Since this is true in each of the 16 subboards, it is true on the entire board.

相似題:一個正方形板, 被平行線分割成36塊長方形, 直橫都有5條線, 每塊面積不必相等, 將長方形黑白相間塗色, 若任意一個白色長方形的面積最多是任意一個黑色格子面積的 3 倍, 請問所有白色格子總面積至多是所有黑色格子總面積的多少倍？

2. Space is dissected into congruent cubes.
Is it necessarily true that for each cube there exists another cube so that both cubes have a whole face in common?

翻譯:空間被分割為許多不重疊的正立方體。
每一個正立方體都存在有另一個正立方體與它有一個完全吻合的面是否必定為真？

3. There are N piles each consisting of a single nut. Two players in turns play the following game. At each move, a player combines two piles that contain coprime numbers of nuts into a new pile. A player who can not make a move, loses. For every N > 2 define which of the players, the first or the second has a winning strategy.

翻譯:有N堆堅果, 每堆都恰好有一顆石子。在一個雙人輪流進行的遊戲中, 每一步:任選石子數量互質的二堆, 合併為一堆, 無法依 再繼續操作的人為輸方。對於N>2，請問玩家先、後手, 有無必勝的策略？

4. Let ABCD be a non-isosceles trapezoid. Define a point A1 as intersection of circumcircle of triangle BCD and line AC. (Choose A1 distinct from C). Points B1, C1, D1 are defined in similar way. Prove that A1B1C1D1 is a trapezoid as well.

翻譯: ABCD 為一非等腰梯型, 對角線 AC 與三角形 BCD 的外接圓另相交於點 A1、與三角形 BAD 的外接圓另相交於點C1;對角線 BD 與三角形 ABC 的外接圓另相交於點D1、與三角形 ADC 的外接圓另相交於點B1。試證四邊形 A1B1C1D1 也有一雙對邊互相平行。

5. In an infinite sequence a1, a2, a3, . . . , the number a1 equals 1, and each an, n > 1, is obtained from an−1 as follows:
- if the greatest odd divisor of n has residue 1 modulo 4, then

$a_n = a_{n-1} + 1$

- and if this residue equals 3, then an = $a_n = a_{n-1} - 1.$
Prove that in this sequence each positive integer occurs infinitely many times.
(The initial terms of this sequence are 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, . . .)

翻譯: 在無窮數列{an}，a0 = 0中，對於n ≥1，若 n 的最大奇因數除以 4 餘數為 1，則 $a_n = a_{n-1} + 1$
;若 n 的最大奇因數除以 4 餘數為 3，則 $a_n = a_{n-1} - 1.$
此數列的首幾項為:0、1、2、1、2、3、2、1、2、3、4、3、2、3、2、1、...。試證在此數列中，每一個正整數將出現無窮多次。

6. Let P(x) be a polynomial with real coefficients so that equation P(m) + P(n) = 0 has infinitely many pairs of integer solutions (m, n). Prove that graph of y = P(x) has a center of symmetry

翻譯:P(x)為實係數多項式, 且存在無窮多的整數對(m, n)使得 P(m) + P(n) = 0。試證它的圖形 y = P(x)具有中心對稱。

7. A test consists of 30 true or false questions. After the test (answering all 30 questions), Victor gets his score: the number of correct answers. Victor is allowed to take the test (the same questions) several times. Can Victor work out a strategy that insure him to get a perfect score after
(a) 30th attempt? (b) 25th attempt?
(Initially, Victor does not know any answer)

翻譯:某項測驗有 30 道是非題。試卷作完後(每一題都必須作答)Victor都會被告知答對的題數, Victor可以對相同的試卷作答好幾次。請問當Victor (a)共作答 30 次 (b)共作答 25 次，Victor有什麼方法保證最後一次作答可以將這 30 道題完全答對？
(Victor對於試題內容完全不懂)

感謝聆聽