International Mathematics TOURNAMENT OF THE TOWNS

Spring 2015

第八組

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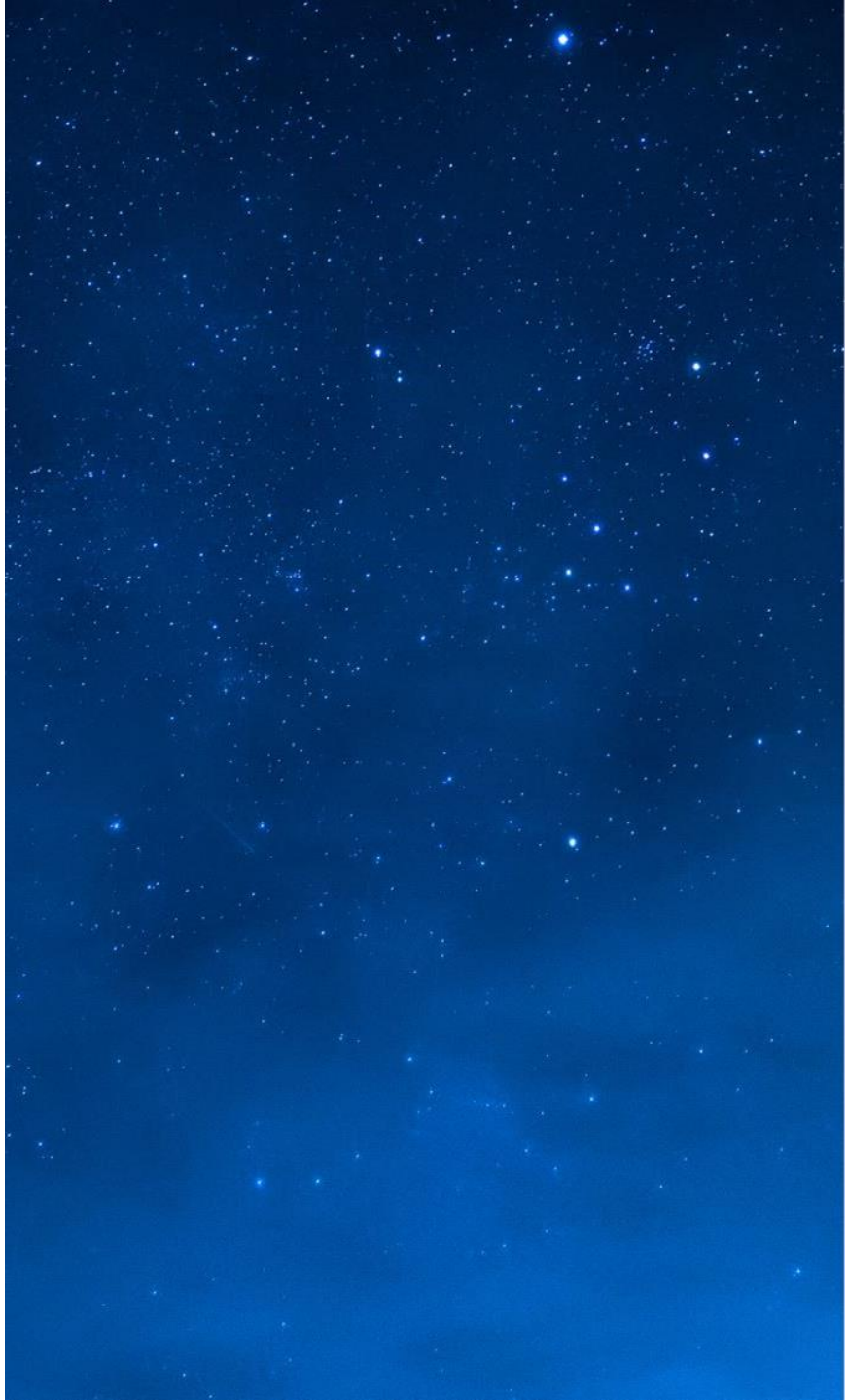

Problem 1.

Question:

(a) The integers x, x^2 and x^3 begin with the same digit. Does it imply that this digit is 1?

(b) The same question for the integers x, x^2 , x^3 , . . . , x^{2015} .

Question:(中文翻譯)

(a)整數x, x^2和x^3以相同的數字開頭

請問這表示開頭的數字為1嗎?

(b)整數$x, x^2, x^3, \ldots, x^{2015}$ 以相同的數字開頭

請問這表示開頭的數字為1嗎?

分析類別:代數

Solution:

(a)

錯誤，反例： $x = 99$, $x^2 = 99^2 = 9801$, $x^3 = 99^3 = 970299$

(b)

由伯努利不等式可知：在$0 \le \varepsilon \le 1$及$k \ge 1$時，有 $(1-\varepsilon)^k \ge 1 - k\varepsilon$ 成立。考慮 $x = 99999$，將其表示成$x = 10^5(1-\varepsilon)$的形式，其中$\varepsilon = 10^{-5}$，那麼$x^k = 10^{5k}(1-\varepsilon)^k \ge 10^{5k}(1-k\varepsilon) \ge 0.9 \times 10^{5k}$。

因此對於所有k=1,2,…,2015， $10^{5k} > x^k \ge 0.9 \times 10^{5k}$，即這些數字皆以9開頭，本題答案為「錯誤」。

伯努利不等式

對實數$x > -1$，

在$n \geq 1$時，有$(1+x)^n \geq 1+nx$成立；

在$0 \leq n \leq 1$時，有$(1+x)^n \leq 1+nx$成立。

等式在n=0,1或是x =0時成立。

可用數學歸納法證明。

伯努利不等式經常用作證明其他不等式的關鍵步驟。

相似題:

整數 $x, x^2, x^3, \ldots, x^n$ 以相同的數字開頭，請問此數字為何?試證明之。

Problem 2.

Question:

A point X is marked on the base BC of an isosceles triangle ABC, and points P and Q are marked on the sides AB and AC so that AP XQ is a parallelogram. Prove that the point Y symmetrical to X with respect to line P Q lies on the circumcircle of the triangle ABC.

Question:(中文翻譯)

在等腰三角形"ABC"中，"BC"邊上有一點"X"

"AB"、"AC"邊上各有點"P"、"Q"

已知"APXQ"為平行四邊形

請證明與"X"以直線"PQ"對稱的點"Y"在"ABC"的外接圓上。

分析類別:幾何

Problem 3.

Question:

(a) A $2 \times n$-table (with $n > 2$) is filled with numbers so that the sums in all the columns are different. Prove that it is possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different.

(b) A 100×100-table is filled with numbers such that the sums in all the columns are different. Is it always possible to permute the numbers in the table so that the sums in the columns would still be different and the sums in the rows would also be different?

Question:(中文翻譯)

(a).A為2xn的矩陣且每行的加總各不相同

請證明交換A中的元素後，各行各列的加總有可能皆不相同。

(b). A為100x100且每行的加總各不相同的矩陣，則是否"所有A"在置換元素過後有可能達成每行每列的加總皆不相同？

分析類別:Matrix(代數)

Problem 4.

Question:

A convex N-gon with equal sides is located inside a circle. Each side is extended in both directions up to the intersection with the circle so that it contains two new segments outside the polygon. Prove that one can paint some of these new 2N segments in red and the rest in blue so that the sum of lengths of all the red segments would be the same as for the blue ones.

Question:(中文翻譯)

有一等長的凸N邊形位於一圓內,現將所有的邊往左右兩個方向擴張到使線段和圓相交.試證明有一方法使2N個線段中左方塗上紅色,剩餘線段為藍色時,紅線的總長和藍線的相等.

分析類別:幾何

Problem 5.

Question:

Do there exist two polynomials with integer coefficients such that each polynomial has a coefficient with an absolute value exceeding 2015 but all coefficients of their product have absolute values not exceeding 1?

Question:(中文翻譯)

請問是否存在兩個整係數多項式,其中兩個多項式中,有一個係數的絕對值大於2015,且其他係數的乘積相乘不超過1

分析類別:代數

Problem 6.

Question:

An Emperor invited 2015 wizards to a festival. Each of the wizards knows who of them is good and who is evil, however the Emperor doesn't know this. A good wizard always tells the truth, while an evil wizard can tell the truth or lie at any moment. The Emperor gives each wizard a card with a single question, maybe different for different wizards, and after that listens to the answers of all wizards which are either "yes" or "no". Having listened to all the answers, the Emperor expels a single wizard through a magic door which shows if this wizard is good or evil. Then the Emperor makes new cards with questions and repeats the procedure with the remaining wizards, and so on. The Emperor may stop at any moment, and after this the Emperor may expel or not expel a wizard. Prove that the Emperor can expel all the evil wizards having expelled at most one good wizard.

Question:(中文翻譯)

有個皇帝邀請2015個術士到一個節慶上，每個術士知道誰是善良的誰是邪惡的，然而國王不知道。善良的術士只說實話，而邪惡的術士會在任何一刻說實話或說假話。國王給每個術士一張有個問題的卡片，可能每個術士的問題都不一樣，然後聽所有術士回答是或否。聽完所有答案之後，國王驅逐一個術士通過一個可以顯示善良或邪惡的魔術門，然後國王製作新問題的卡片，對留下來的術士重複上述的過程。國王可能在任何一刻停止，然後可能驅逐或不驅逐一個術士。

證明國王可以在最多驅逐一位善良術士後驅逐出所有邪惡的術士。

分析類別:邏輯

Problem 7.

Question:

It is well-known that if a quadrilateral has the circumcircle and the incircle with the same centre then it is a square. Is the similar statement true in 3 dimensions: namely, if a cuboid is inscribed into a sphere and circumscribed around a sphere and the centres of the spheres coincide, does it imply that the cuboid is a cube? (A cuboid is a polyhedron with 6 quadrilateral faces such that each vertex belongs to 3 edges.)

Question:(中文翻譯)

眾所周知，如果一個四邊形的外接圓和內接圓同心
那麼它就是正方形。
類似的說法在 3 維上是否成立？
即：
如果一個長方體內接於一個球體並外接於一個球體
並且球體的中心重合，是否意味著該長方體是一個立方體？
（長方體是具有 6 個四邊形面的多面體，每個頂點屬於 3 條邊。）

分析類別:幾何

~感謝聆聽~

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