數學解題方法－作業2
第九組

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1. Each day, the price of the shares of the corporation “Soap Bubble, Limited” either increases or decreases by n percent, where n is an integer such that 0 < n < 100. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

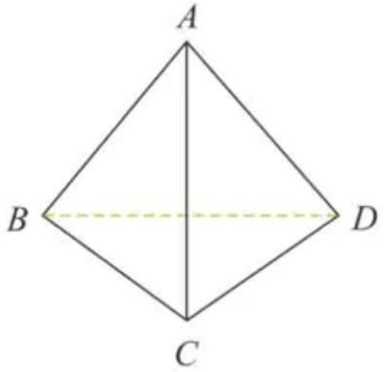
有一家Soap Bubble, Limited公司的股票，它的股價每天不是上漲n%，就是下跌n%，其中0＜n＜100。請問是否存在一個正整數n，使得有二個不同的日期出現相同的股價？

2. All angles of a polygonal billiard table have measures in integral numbers of degrees. A tiny billiard ball rolls out of the vertex A of an interior 1◦ angle and moves inside the billiard table, bouncing off its sides according to the law “angle of reflection equals angle of incidence”. If the ball passes throigh a vertex, it will drop in and stays there. Prove that the ball will never return to A.

一個多邊形(不必是凸的)的撞球枱，它的相鄰兩個邊都互相垂直。在多邊形的頂點處都有進球洞，球洞處掛有網袋 (假想球洞及球為一點)。若頂點A的內角為90°，由A處擊出一顆球，此球在枱內沿直線不停地滾動並依照“入射角等於反射角”之定律碰撞枱邊，球若滾經頂點則會墜入袋中。試證：此球永遠無法回到A袋中。

3. The perpendicular projection of a triangular pyramid on some plane has the largest possible area. Prove that this plane is parallel to either a face or two opposite edges of the pyramid.

一個三角錐(即四面體)在任意平面上的垂直投影中, 已知在平面E上的垂直投影具有最大的面積。試證:平面E平行此三角錐的某一個面, 或與此三角錐的兩條歪斜的稜同時平行。



4. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

在黑板先寫下2004！(= 1×2×3×…×2004)這個數。二人輪流依下列規則玩遊戲。每一次輪到的人都將黑板上的數減掉一個不大於它的自然數，規定這個自然數不同的質因數個數不得超過20個，若所得到結果為0，則其獲勝。否則，擦去黑板上的數，將所得的結果寫在黑板上，再輪到下一位。請問先手或後手何者有必勝的策略？如何操作？

5. The parabola $y = x^2$ intersects a circle at exactly two points A and B. If their tangents at A coincide, must their tangents at B also coincide?

在坐標平面上，拋物線$y = x^2$與圓O恰好只相交於A及B二個點。假設拋物線在點A處的切線與圓在點A處的切線二直線重合。請問拋物線在點B處的切線與圓在點B的切線二直線是否也重合？

解 設圓 O 方程式為 $(x-\alpha)^2+(y-\beta)^2=r^2$，與拋物線 $y=x^2$ 恰有二個交點

$\Rightarrow (x-\alpha)^2+(x^2-\beta)^2-r^2=0$ 恰有兩個解

設 $(x-\alpha)^2+(x^2-\beta)^2-r^2=(x-a)^3(x-b)$，其中 $a\neq b$

$\Rightarrow$ 因左式三次項的係數為 0，所以 $3a+b=0 \Rightarrow b=-3a$

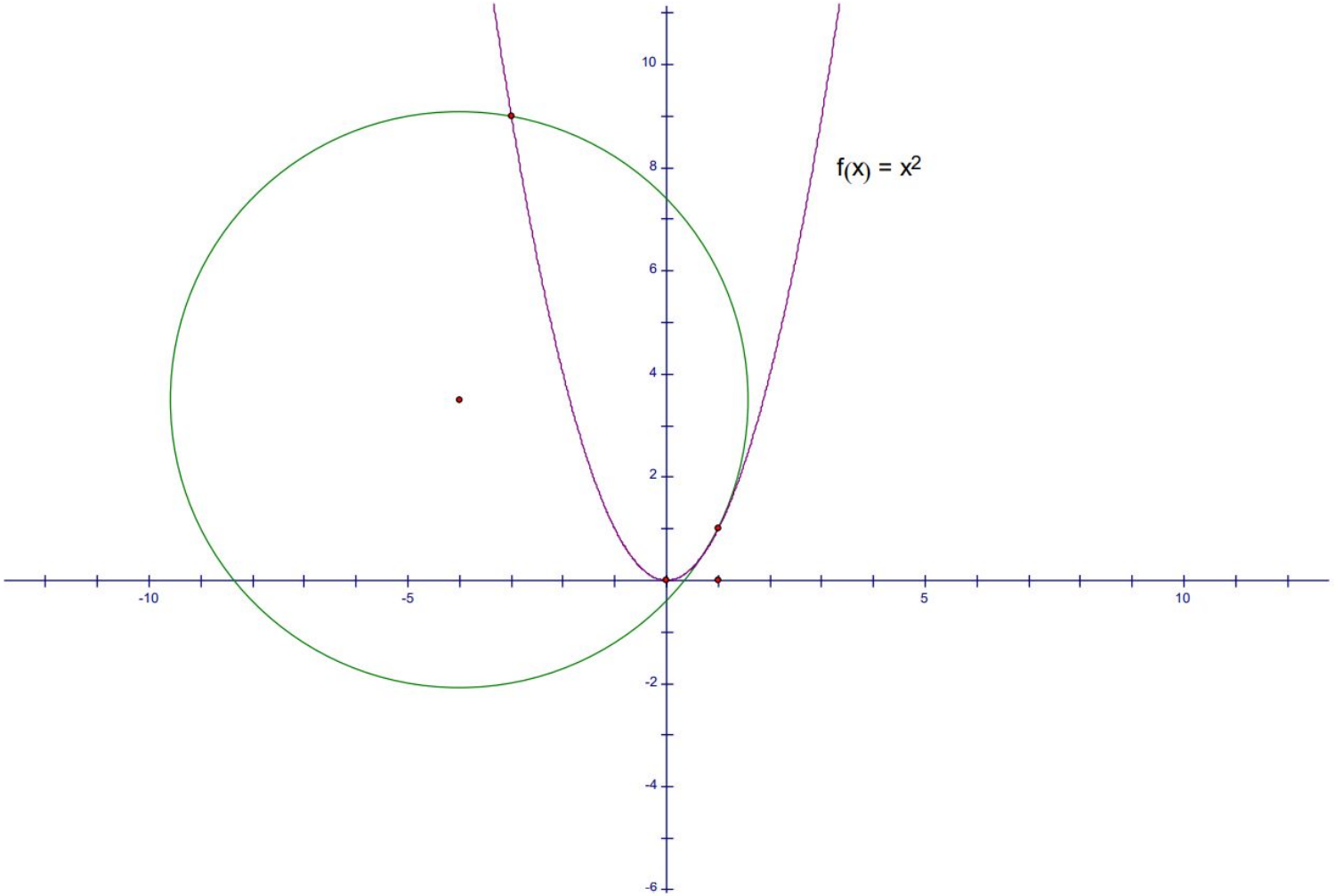
比較兩邊係數得

$$\begin{cases} 1-2\beta=-6a^2 \\ -2\alpha=8a^3 \\ -3a^4=\alpha^2+\beta^2-r^2 \end{cases} \Rightarrow \begin{cases} \alpha=-4a^3 \\ \beta=\dfrac{1+6a^2}{2} \\ r=16a^6+(\dfrac{1+6a^2}{2})^2+3a^4 \end{cases}$$

特例：取 $a=1$，則 $\alpha=4$，$\beta=\frac{7}{2}$，$r=16+\frac{49}{4}+3=\frac{125}{4}$；

代入原方程式求解：$(x-4)^2+(x^2-\frac{7}{2})^2=\frac{125}{4} \Rightarrow x=1、1、1、-3$

$\Rightarrow$ 圓 $(x-4)^2+(y-\frac{7}{2})^2=\frac{125}{4}$ 與拋物線 $y=x^2$ 恰有二個交點，且點 $A(1,1)$ 處，拋物線的切線與圓的切線二直線重合，在點 $B(-3,9)$ 處則否（如下圖）。


f(x) = x2
10
8
6
4
2
-2
-4
-6
-10
-5
5
10

在坐標平面上，拋物線$y = x^2$與橢圓O(已知長軸為5短軸為3，兩焦點所連線平行X軸)恰好只相交於A及B二個點。假設拋物線在點A處的切線與圓在點A處的切線二直線重合。請問拋物線在點B處的切線與圓在點B的切線二直線是否也重合？

解 設圓 O 方程式為

$$\frac{(x-\alpha)^2}{25}+\frac{(y-\beta)^2}{9}=1$$

，與拋物線 $y=x^2$ 恰有二個交點

$\Rightarrow (x-\alpha)^2+(x^2-\beta)^2-225=0$ 恰有兩個解

設 $(x-\alpha)^2+(x^2-\beta)^2-225=(x-a)^3(x-b)$，其中 $a\neq b$

$\Rightarrow$ 因左式三次項的係數為 0，所以 $3a+b=0 \Rightarrow b=-3a$

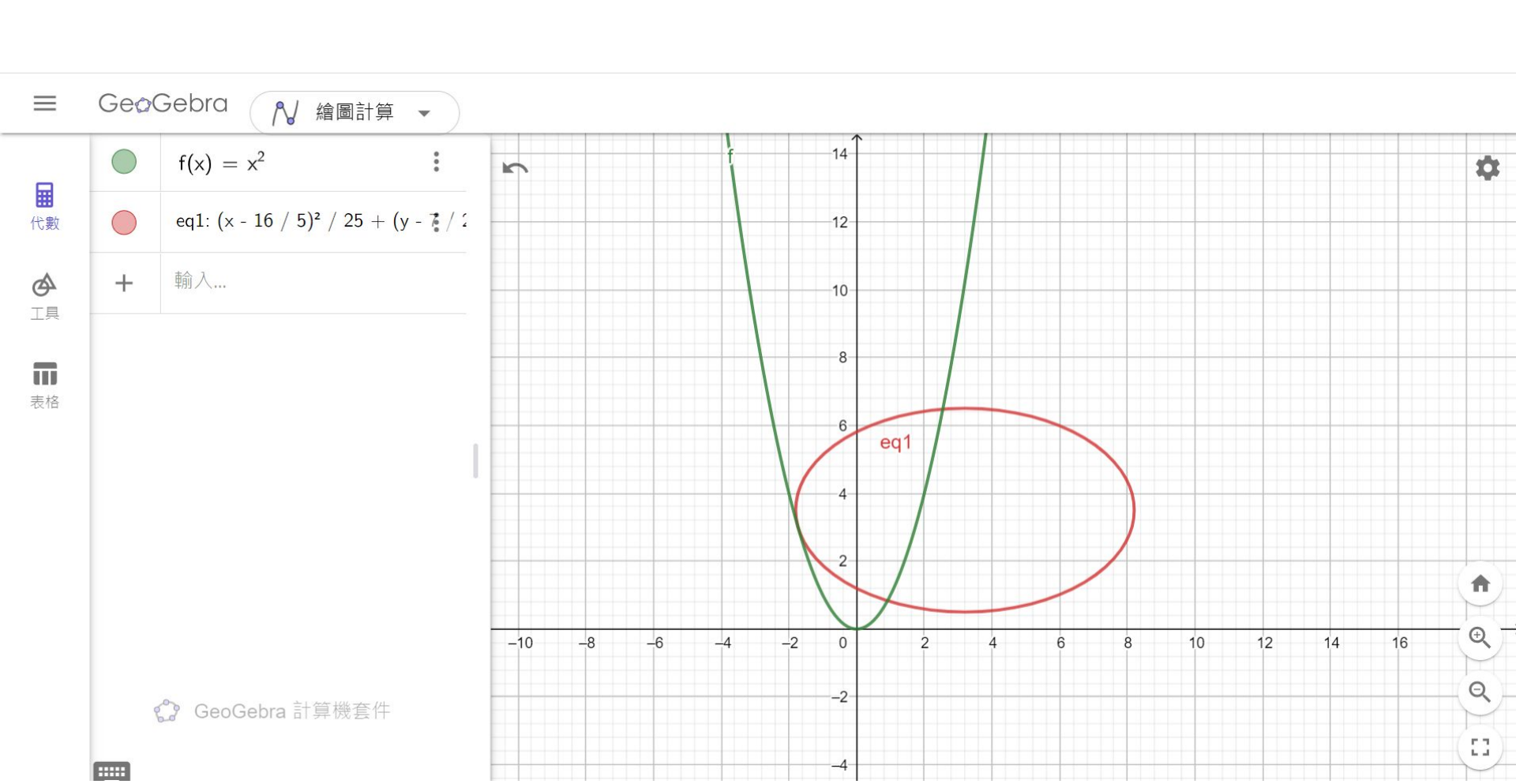
比較兩邊係數得

$$\begin{cases} 1-2\beta=-6a^2 \\ -2\alpha=8a^3 \end{cases} \Rightarrow \begin{cases} \alpha=-4a^3 \\ \beta=\frac{1+6a^2}{2} \end{cases}$$

特例：取 $a=1$，則 $\alpha=16/5$，$\beta=7/2$，；

代入原方程式求解：

$$\frac{(x-\frac{16}{5})^2}{25}+\frac{(y-\frac{7}{2})^2}{9}=1$$


GeoGebra
繪圖計算
f(x) = x²
eq1: (x - 16 / 5)² / 25 + (y -
輸入...
代數
工具
表格
GeoGebra 計算機套件
f
eq1
14
12
10
8
6
4
2
−2
−4
−10
−8
−6
−4
−2
0
2
4
6
8
10
12
14
16

6. The audience shuffles a deck of 36 cards, containing 9 cards in each of the suits spades, hearts, diamonds and clubs. A magician predicts the suit of the cards, one at a time, starting with the uppermost one in the face-down deck. The design on the back of each card is an arrow. An assistant examines the deck without changing the order of the cards, and points the arrow on the back each card either towards or away from the magician, according to some system agreed upon in advance with the magician. Is there such a system which enables the magician to guarantee the correct prediction of the suit of at least (a) 19 cards; (b) 20 cards?

魔法師宣稱他有魔法，可以從撲克牌的背面透視它的花色，於是大家把包括有黑桃、紅心、方塊及梅花各9張的一疊36張牌洗好後，交由助手把這疊牌的牌面向下放在魔法師面前，要他說出這疊牌最上面一張牌的花色。當魔法師說出答案後即把這張牌翻開來驗證他的答案是否正確，並將這張牌放在一旁不再加入這疊牌中；接著繼續猜測下一張牌，重複以上程序。他試圖使猜測的正確次數愈多愈好。這副牌背面的圖樣完全相同但並非對稱的(即可以分辨出圖案朝前或朝後)。助手雖然知道這疊牌的排列順序，但他不能更動，也不能偷偷告訴魔法師。但是他可以依照事先與魔法師約定好的方式擺置撲克牌背面圖案的朝向來暗中協助魔法師。事實上，魔法師並無魔法，他只是利用數學方法來作分析判斷。請問在助手的協助下，魔法師有沒有辦法保證正確地預測到

(a) 不少於19張牌？（三分）

(b) 不少於20張牌？（五分）

感謝聆聽