數學思維與解題-3

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Problem 2 :

Let ABCD be a cyclic quadrilateral, and let diagonals AC and BD intersect at X.
Let C_1, D_1 and M be the midpoints of segments CX, DX and CD, respectively.
Lines AD_1 and BC_1 intersect at Y , and line MY intersects diagonals AC and BD at different points E and F, respectively.
Prove that line XY is tangent to the circle through E, F and X.

令 ABCD 為一個圓內接四邊形，且令對角線 AC 和 BD 相交於 X。令 C_1、D_1 和 M 分別為線段 CX、DX 和 CD 的中點。直線 AD_1 和 BC_1 相交於 Y，直線 MY 與對角線 AC 和 BD 分別相交於不同的點 E 和 F。證明直線 XY 與穿過 E、F 和 X 的圓相切。

Problem 3 :

Let m be a positive integer.
Consider a 4m × 4m array of square unit cells.
Two different cells are related to each other if they are in either the same row or in the same column.
No cell is related to itself.
Some cells are coloured blue, such that every cell is related to at least two blue cells.
Determine the minimum number of blue cells.

令 m 為一個正整數。考慮一個由方形單位單元格組成的 4m × 4m 陣列。如果兩個不同的單元格位於同一行或同一列，則它們彼此相關。單元格不與自身相關。一些單元格被塗成藍色，使得每個單元格都與至少兩個藍色單元格相關。確定藍色單元格的最小數量。

Problem 4 :

Two circles, ω1 and ω2, of equal radius intersect at different points X1 and X2. Consider a circle ω externally tangent to ω1 at a point T1, and internally tangent to ω2 at a point T2. Prove that lines X1T1 and X2T2 intersect at a point lying on ω.

兩個半徑相等的圓 ω1 和 ω2 相交於不同的點 X1 和 X2。考慮一個圓 ω，它在點 T1 處與 ω1 外切，在點 T2 處與 ω2 內切。證明直線 X1T1 和 X2T2 相交於 ω 上的一點。

Problem 5 :

Let k and n be integers such that $k \geq 2$ and $k \leq n \leq 2k - 1$.
Place rectangular tiles, each of size $1 \times k$ or $k \times 1$,
on an $n \times n$ chessboard so that each tile covers exactly
k cells, and no two tiles overlap.
Do this until no further tile can be placed in this way.
For each such k and n, determine the minimum number
of tiles that such an arrangement may contain.

設 k 和 n 為整數，使得 $k \geq 2$ 且 $k \leq n \leq 2k - 1$。在 $n \times n$ 棋盤上放置矩形瓷磚，每個瓷磚的大小為 $1 \times k$ 或 $k \times 1$，使得每個瓷磚恰好覆蓋 k 個單元格，並且沒有兩個瓷磚重疊。一直這樣做，直到無法以這種方式放置更多瓷磚。對於每個這樣的 k 和 n，確定這種排列可能包含的最小瓷磚數。

Problem 6 :

Let S be the set of all positive integers n such that n4 has a divisor in the range n2 + 1, n2 + 2, . . ., n2 + 2n.
Prove that there are infinitely many elements of S of each of the forms 7m, 7m+1, 7m+2, 7m+5, 7m+6 and no elements of S of the form 7m+3 or 7m+4, where m is an integer.

令 S 為所有正整數 n 的集合，使得 n4 在範圍 n2 +1、n2 + 2、...、n2 + 2n 中有一個除數。證明 S 中存在無限多個形式為 7m、7m + 1、7m + 2、7m + 5、7m + 6 的元素，並且 S 中不存在形式為 7m + 3 或 7m + 4 的元素，其中 m 為整數。

Problem 1 :

Let n be an odd positive integer, and let x1, x2, . . ., xn be non-negative real numbers.
Show that

$$\min_{i=1,\ldots,n}(x_i^2 + x_{i+1}^2) \leq \max_{j=1,\ldots,n}(2x_jx_{j+1}),$$

where xn+1 = x1.

令 n 為一個奇數正整數，且令 x1、x2...... xn 為非負實數。
證明其中 $xn_{+1} = x_1$ 。

Solution :

In what follows, indices are reduced modulo n. Consider the n differences $x_{k+1}-x_k$, $k = 1, . . . , n$.

Since n is odd, there exists an index j such that $(x_{j+1} - x_j)(x_{j+2} - x_{j+1}) \geq 0$.

Without loss of generality, we may and will assume both factors non-negative, so $x_j \leq x_{j+1} \leq x_{j+2}$. Consequently,

$$\min_{k=1,\ldots,n}(x_k^2 + x_{k+1}^2) \leq x_j^2 + x_{j+1}^2 \leq 2x_{j+1}^2 \leq 2x_{j+1}x_{j+2} \leq \max_{k=1,\ldots,n}(2x_k x_{k+1}).$$

Problem :

Let n be an even positive integer, and let x1, x2, . . ., xn be non-negative real numbers.
Show that

$$\min_{i=1,\ldots,n}(x_i^2 + x_{i+1}^2) \leq \max_{j=1,\ldots,n}(2x_jx_{j+1}),$$

where xn+1 = x1 does not exist.

令 n 為一個偶數正整數，且令 x1、x2...... xn 為非負實數。
證明其中 xn$_{+1}$ = x$_1$不存在。

Solution:

In what follows, indices are reduced modulo n. Consider the n differences xk+1-xk, k = 1, . . . , n.

for instance, xj = a, and xj+1 = b, where 0 ≤ a < b, so the string of numbers is a, b, a, b, a, . . . , b.

Since n is even, there exists an index j such that (xj+1 - xj)(xj+2 - xj+1) 1 ≤ 0.

(xj+1 -xj)*(xj+2 - xj+1)=(b-a)*(a-b)<0,Hence

$$\min_{k=1,\ldots,n}(x_k^2+x_{k+1}^2)\le x_j^2+x_{j+1}^2\le 2x_{j+1}^2\le 2x_{j+1}x_{j+2}\le \max_{k=1,\ldots,n}(2x_kx_{k+1}).$$

does not exist.

Thank you very much!