# 2019年 EGMO考古題

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# 題目翻譯

# Problem 1

*Day 1. Solutions*

**Problem 1 (Netherlands).** Find all triples $(a, b, c)$ of real numbers such that $ab + bc + ca = 1$ and

$$a^2b + c = b^2c + a = c^2a + b.$$

請找出所有實數(a,b,c)的組合，使得ab+bc+ca=1且a^2b+c=b^c+a=c^2a+b

# Problem 2

**Problem 2 (Luxembourg).** Let $n$ be a positive integer. Dominoes are placed on a $2n \times 2n$ board in such a way that every cell of the board is adjacent to exactly one cell covered by a domino. For each $n$, determine the largest number of dominoes that can be placed in this way.

(A *domino* is a tile of size $2 \times 1$ or $1 \times 2$. Dominoes are placed on the board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. Two cells are said to be *adjacent* if they are different and share a common side.)

設 n 為一正整數。在一個 2n×2n的棋盤上放置骨牌，使得棋盤的每一個格子都恰好與一個被骨牌覆蓋的格子相鄰。對於每個 n，在這樣的放置方式下最多可以放置多少個骨牌。

（骨牌是大小為 2×1或 1×2的方塊。骨牌放置在棋盤上時，每個骨牌恰好覆蓋棋盤上的兩個格子，且骨牌之間不重疊。若兩個格子不同且有共同邊，則稱它們為相鄰。）

# Problem 3

**Problem 3 (Poland).** Let $ABC$ be a triangle such that $\angle CAB > \angle ABC$, and let $I$ be its incentre. Let $D$ be the point on segment $BC$ such that $\angle CAD = \angle ABC$. Let $\omega$ be the circle tangent to $AC$ at $A$ and passing through $I$. Let $X$ be the second point of intersection of $\omega$ and the circumcircle of $ABC$. Prove that the angle bisectors of $\angle DAB$ and $\angle CXB$ intersect at a point on line $BC$.

**三角形ABC的角CAB>角ABC，且I為三角形ABC之內心。使D為線段BC上的一點，使得角CAD=角ABC。使ω為在A點與AC相切並經過I的圓。使X為w與abc外接圓的第二個交點。證明角DAB的角平分線和角CXB交於一點於線段BC上。**

# Problem 4

*Day 2. Solutions*

**Problem 4 (Poland).** Let $ABC$ be a triangle with incentre $I$. The circle through $B$ tangent to $AI$ at $I$ meets side $AB$ again at $P$. The circle through $C$ tangent to $AI$ at $I$ meets side $AC$ again at $Q$. Prove that $PQ$ is tangent to the incircle of $ABC$.

設ABC為一三角形且內心為I。經過點 B 且在 I 點與 AI 相切的圓與邊 AB 相交於點 P 。經過點 C 且在 I 點與 AI 相切的圓與邊 AC 相交於點 Q 。證明 PQ 與 ABC 的內切圓相切。

# Problem 5

**Problem 5 (Netherlands).**

Let $n \geq 2$ be an integer, and let $a_1, a_2, \ldots, a_n$ be positive integers. Show that there exist positive integers $b_1, b_2, \ldots, b_n$ satisfying the following three conditions:

1. $a_i \leq b_i$ for $i = 1, 2, \ldots, n$;
2. the remainders of $b_1, b_2, \ldots, b_n$ on division by $n$ are pairwise different; and
3. $b_1 + \cdots + b_n \leq n \left( \frac{n-1}{2} + \left\lfloor \frac{a_1 + \cdots + a_n}{n} \right\rfloor \right)$.

(Here, $\lfloor x \rfloor$ denotes the integer part of real number $x$, that is, the largest integer that does not exceed $x$.)

**設n>=2為一整數，且令a_1,a_2,...,a_n為正整數。證明存在正整數b_1,b_2,...,b_n滿足以下三個條件：**

**1. 對於i = 1,2,...,n，有a_i <= b_i**

**2. b_1,b_2,...,b_n除以n的餘數不同**

3. $b_1 + \cdots + b_n \leq n \left( \frac{n-1}{2} + \left\lfloor \frac{a_1 + \cdots + a_n}{n} \right\rfloor \right)$

# Problem 6

**Problem 6 (United Kingdom).**

On a circle, Alina draws 2019 chords, the endpoints of which are all different. A point is considered *marked* if it is either

(i) one of the 4038 endpoints of a chord; or

(ii) an intersection point of at least two chords.

Alina labels each marked point. Of the 4038 points meeting criterion (i), Alina labels 2019 points with a 0 and the other 2019 points with a 1. She labels each point meeting criterion (ii) with an arbitrary integer (not necessarily positive).

Along each chord, Alina considers the segments connecting two consecutive marked points. (A chord with $k$ marked points has $k-1$ such segments.) She labels each such segment in yellow with the sum of the labels of its two endpoints and in blue with the absolute value of their difference.

Alina finds that the $N+1$ yellow labels take each value $0, 1, \ldots, N$ exactly once. Show that at least one blue label is a multiple of 3.

(A *chord* is a line segment joining two different points on a circle.)

# Problem 6

在一個圓上，Alina 畫了 2019 條弦，這些弦的端點都不同。如果一個點符合以下任一條件，則該點被視為「標記」：

(i).是 4038 個弦的端點之一

(ii).是至少兩條弦的交點。

Alina 為每個標記點標上標籤。在滿足條件 (i) 的 4038 個點中，Alina 為其中 2019 個點標上 0，另 2019 個點標上 1。她為滿足條件 (ii) 的每個點標上一個任意整數 ( 不一定為正數 ) 。 沿著每條弦，Alina 考慮兩個相鄰標記點之間的線段。（一條具有 k 個標記點的弦有 k - 1 段這樣的線段。）她為每一段這樣的線段標上黃色標籤，其值為該段兩個端點標籤之和，並標上藍色標籤，其值為兩個端點標籤之差的絕對值。 Alina 發現 N + 1 個黃色標籤恰好各取值 0, 1, ..., N 各一次。證明至少有一個藍色標籤是 3 的倍數。（「弦」指的是連接圓上兩個不同點的線段。）

# 講解_Problem 1

## Problem 1

**Find all triples (a,b,c) of real numbers such that $ab+bc+ca=1$ and**

$$a^2 b+c=b^2 c+a=c^2 a+b.$$

## solution

(i) **a,b,c** 其中一個為0

$$ab+bc+ca=1 \Rightarrow ab=1-bc-ca$$

$$\Rightarrow a^2b+c=b^2c+a=a-abc-a^2c+c$$

$$\Rightarrow c(a^2+b^2+ab-1)=0$$

如果$c=0 \Rightarrow ab=1, a^2b=b \Rightarrow a=b=\pm1$

同理，如果$a=0$，得 $b=c=\pm1$。如果$b=0$，得 $a=c=\pm1$

因此，如果 **a,b,c**其中一個=0，另外兩個分別為**+1**和**-1**，就可以滿足條件。

(ii) **a,b,c** 均不為0

**$a^2+b^2+ab=1$ –(1)**

**$b^2+c^2+bc=1$ –(2)**

**$c^2+a^2+ca=1$ –(3)**

**(1)+(2)+(3)$\Rightarrow 2(a^2+b^2+c^2)+(ab+bc+ca)=3$**

**$\Rightarrow (a^2+b^2+c^2)=1$**

**$\Rightarrow (a+b)^2+(b+c)^2+(c+a)^2=0$**

**$\Rightarrow a=b=c$** 代入**$a^2+b^2+ab=1$**

**$\Rightarrow a=b=c=\pm\sqrt{1/3}$**

因此，如果 **a,b,c**均不=0，**(a,b,c)=±(√1/3,√1/3,√1/3)**，就可以滿足條件。

# 相似的Problem1

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**Find all triples (a,b,c) of real numbers such that $ab+bc+ca=4$**

**and**

$$a^2 b+4c=b^2 c+4a=c^2 a+4b.$$

# solution

(i) **a,b,c** 其中一個為0

$ab+bc+ca=4 \Rightarrow ab=4-bc-ca$

$\Rightarrow a^2 b+4c=b^2 c+4a=c^2 a+4b$

如果$c=0 \Rightarrow ab=1, a^2 b=b \Rightarrow a=b=\pm 2$

同理，如果$a=0$，得 $b=c=\pm 2$ 。如果$b=0$，得 $a=c=\pm 2$

因此，如果 **a,b,c**其中一個=0，另外兩個分別為**+2**和**-2**，就可以滿足條件。

## (ii) a,b,c 均不為0

$a^2+b^2+ab=4$ –(1)

$b^2+c^2+bc=4$ –(2)

$c^2+a^2+ca=4$ –(3)

(1)+(2)+(3)$\Rightarrow 2(a^2+b^2+c^2)+(ab+bc+ca)=12$

$\Rightarrow a^2+b^2+c^2=4$

$\Rightarrow a^2+ab+b^2=4$

$\Rightarrow c^2=ab$

同理 $\Rightarrow b^2=ca$ & $a^2=bc$

$\Rightarrow a=b=c$ 代入$a^2+b^2+ab=4$

$\Rightarrow a=b=c=\pm 2/\sqrt{3}$

因此，如果 a,b,c均不=0，$(a,b,c)=\pm(2/\sqrt{3},2/\sqrt{3},2/\sqrt{3})$